Fall term 2026 Prof. Andreas Mihatsch
Linear algebraic groups

6th Homework (due Oct 28)

k always denotes an algebraically closed field of characteristic 0 and varieties are over k.

Problem 1 (Vector fields on SL_2)

We discussed in class that $\mathfrak{sl}_2 = \{x \in \mathfrak{gl}_2 \mid \mathrm{tr}(x) = 0\}$.

(a) For a choice of basis vectors of $\mathfrak{sl}_2$, calculate their extensions to translation invariant vector fields on SL_2.

(b) Calculate the commutators of these three vector fields and, in this way, derive a formula for the Lie bracket on $\mathfrak{sl}_2$.

Remark: The result will simply be the standard commutator bracket of matrices, but the point of the exercise is to also gain some intuition for invariant vector fields.

Problem 2 (Vector fields on homogeneous spaces)

Let G act transitively on X. That is, $X = G/G_x$ when $x \in X$.

(a) For simplicity, assume that X is affine. Define what it means for a vector field on X to be G-invariant. Prove that there is an isomorphism

$$\mathcal{T}(X)^G \xrightarrow{\sim} \mathcal{T}_x(X)^{G_x}$$

from G-invariant vector fields on X to the G_x-invariants of the tangent space at x.

(b) Consider $G = \mathrm{GL}_n$ acting naturally on $X = \mathbb{P}^{n-1}$. Let $x = [1 : 0 : \ldots : 0] \in \mathbb{P}^{n-1}$ with stabilizer G_x. Determine G_x and its action on $\mathcal{T}_x(\mathbb{P}^{n-1})$. Determine the dimension of

$$\mathcal{T}_x(\mathbb{P}^{n-1})^{G_x}.$$

Remark: This determines the space of GL_n-invariant vector fields on $\mathbb{P}^{n-1}$. (How would one define this notion?)

Problem 3 (3-Dimensional semi-simple Lie algebras)

Let $(\mathfrak{g}, [\ ,\ ])$ be a 3-dimensional Lie algebra. Assume that $\mathfrak{g}$ has no proper *ideals*, i.e. proper subspaces $\mathfrak{h} \subset \mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$. Show that $\mathfrak{g}$ is already isomorphic to $\mathfrak{sl}_2$. That is, it has a basis e, f, h with

$$[e, f] = h,\ [e, h] = -2e,\ [f, h] = 2f.$$

Problem 4 (Dual numbers)

Let $X = \mathrm{Sp}(A)$ be an affine variety and let $x \in X$ correspond to $\mathfrak{m} \subset A$. Construct isomorphisms

$$\begin{aligned} \mathcal{T}_x(X) &:= \mathrm{Der}_k(A, A/\mathfrak{m}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{k\text{-alg}}(A/\mathfrak{m}^2, k[\varepsilon]/(\varepsilon^2)) \\ &\xrightarrow{\sim} \mathrm{Hom}_{k\text{-vect}}(\mathfrak{m}/\mathfrak{m}^2, k). \end{aligned}$$