Fall term 2026 Prof. Andreas Mihatsch
Linear algebraic groups

5th Homework (due Oct 21)

k always denotes an algebraically closed field of characteristic 0 and varieties are over k.

Problem 1 (Characteristic polynomials for SO_n and Sp_{2m})

Let $(\ ,\ ) : V \times V \to k$ be a non-degenerate symmetric or alternating pairing on a finite-dimensional k-vector space V. Let $G = \mathrm{SO}(V, (\ ,\ ))$ resp. $G = \mathrm{Sp}(V, (\ ,\ ))$ be the attached special orthogonal or symplectic group.

Prove that $\mathrm{Hom}_{\mathrm{LAG}}(G, \mathbb{G}_a) = 0$.

(a) Let $g \in G$ be an element and let $V = \bigoplus_{\lambda \in k^\times} V_\lambda$ be the generalized eigenspace decomposition of V. Show that if $\lambda\mu \neq 1$, then

$$(v, w) = 0 \quad \text{for all} \quad v \in V_\lambda,\ w \in V_\mu.$$

Deduce that for $\lambda \in k^\times$, $(\ ,\ )$ puts V_λ and $V_{\lambda^{-1}}$ into perfect duality.

(b) Prove that the characteristic polynomial $P(t)$ of an element $g \in G$ has the symmetry

$$P(t) = (-1)^n t^n P(1/t).$$

(c) View $\mathbb{A}^n$ as the space of monic degree n polynomials by recording their coefficients:

$$(a_{n-1}, \ldots, a_0) \longmapsto t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0.$$

Show that the subset $X = \{P(t) \in \mathbb{A}^n \mid (-1)^n t^n P(1/t) = P(t)\}$ is an irreducible closed subvariety and that the following morphism is surjective:

$$G \longrightarrow X, \quad g \longmapsto \mathrm{char}(g; t).$$

Problem 2 ($\mathrm{Hom}(\mathrm{SO}_n, \mathbb{G}_a)$ and $\mathrm{Hom}(\mathrm{Sp}_{2m}, \mathbb{G}_a)$ vanish)

(a) With G as above, assume that $g \in G$ is diagonalizable with different eigenvalues. Show that the centralizer $\mathrm{Cent}(g) \subseteq G$ is isomorphic to $(\mathbb{G}_m)^{\lfloor \dim(V)/2 \rfloor}$.

(b) Prove that a dense subset of elements of G is diagonalizable with different eigenvalues. Show that $\mathrm{Hom}(G, \mathbb{G}_a) = 0$.

**** Pick one of the following according to taste ****

Problem 3a (GL_2-orbits on a projective variety)

Let $G = \mathrm{GL}_2$ act on $\mathbb{P}^1$ by the standard action $g \cdot \ell = g(\ell)$.

How many G-orbits are there for the diagonal G-actions on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$? That is, G acts by

$$g \cdot (\ell_1, \ell_2) = (g(\ell_1), g(\ell_2)), \quad g \cdot (\ell_1, \ell_2, \ell_3) = (g(\ell_1), g(\ell_2), g(\ell_3)).$$

Problem 3b (When is $(B \otimes_A C)_{\text{red}} = B \otimes_A C$?)

Let A, B and C be reduced finite type k-algebras and let $A \to B$ and $A \to C$ be k-algebra maps. The goal of the problem is to show that if B and C are flat over A, then $B \otimes_A C$ is reduced. In particular,

$$\text{Sp}(B) \times_{\text{Sp}(A)} \text{Sp}(C) = \text{Sp}(B \otimes_A C) \quad \text{on the nose!}$$

For simplicity, we assume A, B and C are integral domains.

(a) Use flatness to embed $B \otimes_A C$ into

$$\text{Quot}(B) \otimes_{\text{Quot}(A)} \text{Quot}(C).$$

(b) Prove that a tensor product $L_1 \otimes_K L_2$ of field extensions in characteristic 0 is reduced. (You can assume L_1, L_2 finitely generated over K.)

$\longrightarrow$ This shows that $B \otimes_A C$ is reduced.