Fall term 2026 — Prof. Andreas Mihatsch

Linear algebraic groups

Friendly reminder:

Homework problems are meant to help studying the lecture contents and will not count towards final grade. Points are given for feedback only

The final grade will be formed from the mid-term exam (Nov 4, 40%) and the final exam (Dec 30, 60%)

2nd Homework sheet (due Sept 30)

k always denotes an algebraically closed field of characteristic 0.

Problem 1 (Topological properties, 3P)

Let $f : X \to Y$ be a continuous map of topological spaces. Show that if X has one of the properties quasi-compact/connected/irreducible, then $f(X)$ has the same property.

Problem 2 (Irreducible components algebraically, 4P)

Let A be a reduced finite type k-algebra. Show that the intersection $\bigcap_{i\in I} \mathfrak{p}_i$ of a set of prime ideals is a radical ideal. Conversely, show that a radical ideal $\mathfrak{a} \subseteq A$ equals the intersection $\bigcap_{\mathfrak{a}\subseteq\mathfrak{p}} \mathfrak{p}$ of all prime ideals it is contained in.

Throughout your answer, be precise about how you use Hilbert's Nullstellensatz.

Problem 3 (Centralizers, 3P + 3P)

(a) Let G be a LAG over k and let $x \in G$ be an element. Show that the centralizer

$$G_x := \{g \in G \mid gx = xg\}$$

is a closed subgroup of G.

Hint: First show that the diagonal $\Delta \subseteq X \times X$ is closed for any $X = \mathrm{Sp}(A)$. Then write G_x as an inverse image of Δ.

(b) Determine the centralizers of

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in \mathrm{SL}_2 \qquad \text{and} \qquad \begin{pmatrix} 2 & & \\ & 2 & \\ & & 1 \end{pmatrix} \in \mathrm{GL}_3\,.$$

Problem 4 (Centers, 2P + 3P)

(a) Building on the previous exercise, show that the center of an algebraic group is a closed subgroup.

(b) Determine the center of the subgroup of upper triangular matrices $T_n \subseteq \mathrm{GL}_n$. Write it as a product of copies of $\mathbb{G}_m$ and $\mathbb{G}_a$.