Fall term 2026 Prof. Andreas Mihatsch
Linear algebraic groups

Friendly reminder:

Homework problems are meant to help studying the lecture contents and will not count towards final grade. Points are given for feedback only

The final grade will be formed from the mid-term exam (Nov 4, 40%) and the final exam (Dec 30, 60%)

1st Homework sheet (due Sept 23)

k always denotes an algebraically closed field of characteristic 0

Problem 1 (Principal open subsets are again affine varieties, 5P)

Let A be a reduced finite type k-algebra, let $X = \mathrm{Sp}(A)$ and let $f \in A$.

Show that the non-vanishing set $D(f) \subseteq X$ is again naturally an affine variety. More precisely, use the natural map $A \to A[f^{-1}]$ to construct a map

$$\mathrm{Sp}(A[f^{-1}]) \longrightarrow D(f),$$

prove that it is a homeomorphism, and show that for every open $V \subseteq D(f)$ also $\mathcal{O}_{\mathrm{Sp}(A)}(V) = \mathcal{O}_{\mathrm{Sp}(A[f^{-1}])}(V)$.

Problem 2 (Closed subsets are again affine varieties, 3P)

Let A be a reduced finite type k-algebra, let $X = \mathrm{Sp}(A)$ and let $Z \subseteq X$ be a closed subset. Let $I = I(Z) \subseteq A$ be the corresponding ideal (Hilbert's Nullstellensatz).

Show that Z is again naturally an affine variety. Concretely, show that the natural map $A \to A/I$ induces a homeomorphism

$$\mathrm{Sp}(A/I) \longrightarrow Z.$$

(This allows to *define* the variety structure on Z as that of $\mathrm{Sp}(A/I)$, so there is nothing further to check.)

Problem 3 (The special orthogonal group SO_2, 5P)

Let $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and let $\mathrm{SO}_2 = \mathrm{SO}(H)$ be the special orthogonal group.

Prove that $\mathrm{SO}_2 \xrightarrow{\sim} \mathbb{G}_m$ as algebraic group.

Hint: It was exlained during lecture how to define $\mathrm{SO}(H)$ *as vanishing set of equations.*

Problem 4 (Endomorphisms of $\mathbb{G}_a$, 2P + 3P)

(a) Describe the set of self-morphisms $\mathbb{A}^1 \to \mathbb{A}^1$ and its composition law.

(b) Determine the endomorphism ring $\mathrm{End}(\mathbb{G}_a)$, i.e. all those $f : \mathbb{A}^1 \to \mathbb{A}^1$ that satisfy $f(x + y) = f(x) + f(y)$.