LINEAR ALGEBRAIC GROUPS

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ABSTRACT. These are notes for a graduate course on linear algebraic groups taught at Zhejiang University. They were written by Zhu Ruizhi during class for which I thank him sincerely. Any comments are highly appreciated; please send them directly to mihatsch@zju.edu.cn.

CONTENTS

1. INTRODUCTION (9/16/2026)

Course Information

- Prerequisites: Commutative Algebra or Introduction to Algebraic Geometry
- One lecture per week, on Wednesdays 10:00 a.m. to 12:30 p.m.
- Homework graded for feedback, no effect on final grade
- Two exams:
  - 1st on November 4th, 40%.
  - 2nd on December 30th, 60%.
- Main reference: Humphrey's Linear Algebraic Groups [1]

The goal of this first lecture are to explain what the theory of linear algebraic groups is about, to introduce affine varieties, and to give the definition of a linear algebraic group.

1.1. **What is the theory about?**
Mathematicians and physicist are interested in groups such as GL_n, SL_n, SO_n, or Sp_{2n}. These arise in many different contexts and with different flavors:

(1) Finite groups of Lie type $GL_n(\mathbb{F}_q), Sp_{2n}(\mathbb{F}_q)$,... come from a more set-theoretic, combinatorial perspective. They make up part of classification of the finite simple groups.
(2) Real or complex Lie groups $GL_n(\mathbb{R}), SO_n(\mathbb{C}), SL_n(\mathbb{R}), PGL_n(\mathbb{C})$,... arise from a more analytic perspective which is relevant in differential geometry, representation theory, etc.

Our perspective. We want to study these groups as algebraic objects. An advantage of this approach is that it makes sense over various fields. For example, we can study $SO(V)/\mathbb{Q}, GL_n/\mathbb{Q}_p, Sp_{2n}/\mathbb{C}$ which is clearly relevant for areas such as number theory, commutative algebra, algebraic geometry, and so on.

Date: Winter term 2026.

Example 1.1. We consider $SL_2/\mathbb{C}$ as algebraic or complex Lie group, and are interested in its representation theory.

By definition, a representation of SL_2 is a finite-dimensional $\mathbb{C}$-vector space V together with a homomorphism $SL_2 \to GL(V)$. Such representations can be completely classified:

Theorem 1.2. *Let V be the finite-dimensional representation of* $\mathrm{SL}_2/\mathbb{C}$.

(a) V is completely reducible, *meaning it is a direct sum of irreducible representations,*

$$V \cong \bigoplus_{i=1}^{r} V_i.$$

(b) Each irreducible summand V_i is isomorphic to the n-th symmetric power $\mathrm{Sym}^n(\mathbb{C}^2)$ *of the standard representation, for some n.*

Explanation. What is $\mathrm{Sym}^n(\mathbb{C}^2)$?

(1) $SL_2 \curvearrowright \mathbb{C}^2$ is the *standard representation*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}. \tag{1.1}$$

(2) $\mathrm{Sym}^n(\mathbb{C}^2)$ is derived from $\mathbb{C}^2$ as the space of homogeneous degree n polynomials in x and y:

$$\mathrm{Sym}^n(\mathbb{C}^2) = \{a_n x^n + a_{n-1}x^{n-1}y + \cdots + a_0 y^n \mid a_n, \dots, a_0 \in \mathbb{C}\}.$$

SL_2 acts by the coordinate substitution given by (1.1):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot p(x,y) = p(ax+by, cx+dy).$$

For example,

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \cdot x^2 = (x+y)^2 = x^2 + 2xy + y^2.$$

Theorem 1.2 is a beautiful example of a complete classification result. But such behavior is not automatic for algebraic groups, as the next example shows.

Example 1.3. Consider the *additive group* $\mathbb{G}_a = (\mathbb{C}, +)$. It has the two-dimensional representation

$$\mathbb{G}_a \longrightarrow GL_2, \qquad t \longmapsto \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}.$$

- This representation is not irreducible: $\mathbb{C} \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \subseteq \mathbb{C}^2$ is a $\mathbb{G}_a$-stable subspace.
- But $\mathbb{C} \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ has no invariant complement:

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ b \end{pmatrix} \right\rangle = \mathbb{C}^2 \qquad \text{if } b \neq 0.$$

Question. Which groups behave like SL_2, which like $\mathbb{G}_a$?

General theory answers this question:

- *reductive groups* behave like SL_2 (completely reducible representation theory)
- *unipotent groups* behave like $\mathbb{G}_a$ (every rep has a fixed vector like $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \in \mathbb{C}^2$ before)

Example 1.4. The groups $GL_n, SL_n, PGL_n, SO_n, Sp_{2n}$ are all reductive.

Upper triangular matrices form a unipotent group, e.g.

$$\left\{\begin{pmatrix}1 & x & y\\ & 1 & z\\ & & 1\end{pmatrix}\right\} \subseteq GL_3.$$

Question. OK, but I've already seen all these groups before. Can you teach me some new ones?

Answer. General theory says that these examples are the building blocks for *all* LAGs!

Theorem 1.5. *Let $G/\mathbb{C}$ be a connected linear algebraic group.*

(a) There exists a unique maximal, normal, unipotent subgroup $G_u \subseteq G$ called the unipotent radical *of G. The quotient $G_{\text{red}} = G/G_u$ is reductive.*

(b) The unipotent part G_u is built as a successive extension of copies of $\mathbb{G}_a$. The reductive quotient G_{red} is built out of GL_1, SL_n, SO_n and Sp_{2n}.

Example 1.6. Consider the block upper triangular subgroup

$$G := \left\{\begin{pmatrix}x_{11} & x_{12} & y\\ x_{21} & x_{22} & z\\ 0 & 0 & x_{33}\end{pmatrix}\right\} \subseteq GL_3.$$

Then

$$G_u = \left\{\begin{pmatrix}1 & 0 & y\\ & 1 & z\\ & & 1\end{pmatrix}\right\} \cong \mathbb{G}_a^2, \qquad G_{\text{red}} \cong GL_2 \times GL_1.$$

Example 1.7. The *projective linear group PGL_n* is reductive and obtained as quotient

$$SL_n \xrightarrow{n:1} PGL_n.$$

The *spinor group* Spin_n is reductive and obtained as a double cover

$$\mathrm{Spin}_n \xrightarrow{2:1} SO_n, \qquad n \geq 3.$$

The general theory also more precisely classifies all reductive groups and their representations up to isomorphism, which we will see towards the end of the course.

To end this introduction, let us consider a different type of statement.

Question. How to work with group elements "like a professional"?

Example 1.8. Any $g \in GL_n(\mathbb{C})$ can be written uniquely as a product $g = g_s \cdot g_u$ such that

- g_s is diagonalizable (also called *semi-simple*)
- g_u is unipotent (meaning $g_u - 1 \in M_n(\mathbb{C})$ is nilpotent)
- $g_s \cdot g_u = g_u \cdot g_s$

Indeed, we know that g has a unique *Jordan normal form* up to permutation of blocks, and the decomposition $g = g_s \cdot g_u$ is immediate to see in JNF. For example,

$$g = \begin{pmatrix}2 & 5 & \\ & 2 & \\ & & -1\end{pmatrix} = \underbrace{\begin{pmatrix}2 & & \\ & 2 & \\ & & -1\end{pmatrix}}_{g_s} \cdot \underbrace{\begin{pmatrix}1 & 5/2 & \\ & 1 & \\ & & -1\end{pmatrix}}_{g_u}.$$

Here, note that g_s and g_u indeed commute as required.

Theorem 1.9. *Let G be a connected LAG over $\mathbb{C}$. Any $g \in G$ has a unique decomposition $g = g_s g_u$ into a semi-simple and unipotent part, characterized by*

$$\forall\ \rho : G \longrightarrow GL_n, \qquad \rho(g) = \rho(g_s) \cdot \rho(g_u)$$

is the decomposition from Example 1.8 $\rho(g)$.

Conclusion. Summing up this lecture, we have seen that the theory of LAGs

- Classifies and describes algebraic groups such as $GL_n, SL_n, \mathrm{Spin}_n$,
- Classifies and describes their representations, and
- Provides structural tools for working with such groups or their group elements.

1.2. **Affine varieties.** Reference: [1, §1.1-1.5]

In order to define linear algebraic groups, we first need to explain how to capture the idea of an *algebraic* object. This is done by the notion of an *affine algebraic variety.*

Let k be an algebraically closed field.

Slogan. An affine variety over k is the vanishing set in some k^n defined by a family of polynomial equations.

We now make this definition precise. Throughout, let

$$A \cong k[T_1, \dots, T_n]/(f_1, \dots, f_m)$$

be a reduced finite type k-algebra. We want to define the affine variety $\mathrm{Sp}(A)$ attached to A, which will be done in three steps.

1st Step: The underlying set. Define the *Max-Spectrum* of A as

$$\mathrm{Sp}(A) := \{\text{maximal ideals of } A\}.$$

Because k is assumed algebraically closed, for every $\mathfrak{m} \in \mathrm{Sp}(A)$, the residue field of $\mathfrak{m}$ is naturally identified with k:

$$\begin{tikzcd} k \arrow[r] \arrow[dr, "\cong"'] & A \arrow[d, "\text{quotient}"] \\ & A/\mathfrak{m} \end{tikzcd} \qquad \text{(also use } A \text{ is a } k\text{-algebra.)}$$

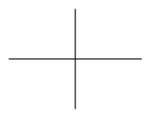
So we see:

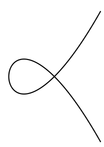
$$\begin{aligned} \mathrm{Sp}(A) &= \{k\text{-alg homom } A \longrightarrow k\} \\ &= \{(x_1, \dots, x_n) \in k^n \mid f_1(x_1, \dots, x_n) = \cdots = f_m(x_1, \dots, x_n) = 0\}. \end{aligned}$$

choosing presentation $A = k[T_1, \dots, T_n]/(f_1, \dots, f_m)$

Example 1.10.

$$\mathrm{Sp}(k[T_1, \dots, T_n]) \ = \ k^n$$

$\mathrm{Sp}(k[X,Y]/(XY)) \ =$  $\subseteq k^2$

$\mathrm{Sp}(k[X,Y]/(Y^2 - X^2(X+1))) \ =$  $\subseteq k^2$

2nd Step: Upgrade to a toplogical space. Every $f \in A$ defines a function $\mathrm{Sp}(A) \longrightarrow k$

$$\begin{tikzcd} k \arrow[r] \arrow[dr, equal] & A \arrow[d, "\mathfrak{m} \in \mathrm{Sp}(A)"] \\ & k \end{tikzcd} \qquad \begin{array}{c} f \\ \downarrow \\ f \bmod \mathfrak{m}. \end{array}$$

We usually write $x = \mathfrak{m} \in \mathrm{Sp}(A)$ and $f(x) := f \bmod \mathfrak{m} \in k$ where $k = A/\mathfrak{m}$, because this is much more intuitive notation.

Polynomial functions should definitely be continuous, but k has no topology. So the best one can do in algebraic geometry is to *define* open and closed subsets by

$$\begin{aligned} V(f) &= \{x \in \mathrm{Sp}(A) \mid f(x) = 0\} && \text{closed "vanishing set"} \\ D(f) &= \{x \in \mathrm{Sp}(A) \mid f(x) \neq 0\} && \text{open "doesn't vanish set".} \end{aligned} \tag{1.2}$$

Definition 1.11. Endow $\mathrm{Sp}(A)$ with topology generated by $\{D(f)\}_{f \in A}$.

What does this mean concretely? We already have the following opens from (1.2):

$$\mathrm{Sp}(A) = D(1), \qquad \varnothing = D(0)$$

$$D(f_1) \cap \cdots \cap D(f_r) = D(f_1 \cdots f_r)$$

So the potential new open subsets are the arbitrary unions $\bigcup_{i \in I} D(f_i)$ of sets of the form $D(f)$. Note that an open of the form $D(f)$ is called *principal open subset.*

But A is noetherian: Ther exist finitely many $g_1, \ldots, g_r$ generating the ideal $(f_i)_{i \in I}$. Then

$$\bigcap_{i \in I} V(f_i) = \bigcap_{j=1}^{r} V(g_j) \quad \text{or equivalently,} \quad \bigcup_{i \in I} D(f_i) = \bigcup_{j=1}^{r} D(g_j).$$

Conclusion. The open subsets of $\mathrm{Sp}(A)$ are precisely the finite unions $\bigcup_{i=1}^{r} D(f_i)$.

Dually, the closed subsets are precisely the finite intersections

$$V(f_1, \ldots, f_r) := \bigcap_{i=1}^{r} V(f_i).$$

Next, we want to connect this topology closer to the ring A. We in fact have back-and-forth definitions:

$$\begin{aligned} A \supseteq S \text{ subset} \ &\longmapsto \ V(S) := \bigcap_{s \in S} V(s), \\ I(Z) := \{f \in A \mid f(z) = 0 \ \forall z \in Z\} \ &\longleftarrow\!\!\dashv \ Z \subseteq \mathrm{Sp}(A) \text{ closed.} \end{aligned}$$

You should convince yourself of the following easy properties:

- $V(S)$ is closed
- $I(Z)$ an ideal
- $V(S) = V((S))$, $(S) \subseteq A$ ideal generated by S
- If $f^r \in I(Z)$, then $f \in I(Z)$. That is, $I(Z)$ is a so-called *radical* ideal. For example, $V(f^r) = V(f)$.

Theorem 1.12 (Hilbert's Nullstellensatz)**.** *The above maps define a bijection*

$$\{\textit{radical } I \subseteq A\} \xleftrightarrow{1:1} \{\textit{closed } Z \subseteq \mathrm{Sp}(A)\}$$

$$I \longmapsto V(I),$$

$$I(Z) \longleftarrow\!\!\dashv Z.$$

The significance of this theorem is that it allows to translate any statement about unions, intersections, non-emptyness etc. of closed or open subsets of $\mathrm{Sp}(A)$ into a statement about ideals in A, and vice versa. That is, it provides a dictionary between geometry and algebra.

3rd Step: Equip $\mathrm{Sp}(A)$ with its polynomial functions. Set $X = \mathrm{Sp}(A)$ in the following. For every open $U \subseteq X$, we want to say what the polynomial functions on U are. This is similar to the definition of the smooth functions on an open subset of a manifold, for example.

For traditional reasons, what we intuitively just called "polynomial" functions, is usually called *regular functions*.

Definition 1.13. For $U \subseteq \mathrm{Sp}(A)$ open, we denote by $\mathscr{O}_X(U)$ the ring of all regular functions $f : U \to k$.

(1) If $U = D(h)$ is principal open, then by definition

$$\mathscr{O}_X(D(h)) := A[h^{-1}].$$

Note that this makes sense since $h : D(h) \to k^\times$ is an invertible function.

(2) For general U, we say that $f : U \to k$ is regular if $\forall D(h) \subseteq U$, the restriction $f|_{D(h)} \in \mathscr{O}_X(D(h))$.

Definition 1.14. The pair $(\mathrm{Sp}(A), \mathscr{O}_{\mathrm{Sp}(A)})$ is the *affine variety* defined by A. Here, $\mathrm{Sp}(A)$ denotes the topological space, and $\mathscr{O}_{\mathrm{Sp}(A)}$ denotes the regular functions on $\mathrm{Sp}(A)$. But usually one only writes $\mathrm{Sp}(A)$ for brevity.

Example 1.15. The *n-dimensional affine space* over k is

$$\mathbb{A}^n := \mathrm{Sp}\, k[T_1, \dots, T_n].$$

(1) Underlying set $= k^n$,
(2) Opens $=$ complements of vanishing sets $V(f_1, \dots, f_m)$,
(3) Regular functions $=$

$$\mathscr{O}_{\mathbb{A}^n}(U) = \left\{ \frac{p}{q} \in k(T_1, \dots, T_n) \;\middle|\; q(x) \neq 0 \;\forall x \in U \right\}.$$

Example 1.16. Here is an interesting example of regular functions on a non-principal open: Consider $\mathbb{A}^2 = \mathrm{Sp}\, k[X, Y]$ with open subset $U = k^2 \setminus \{(0,0)\}$. This is the union of $D(X)$ and $D(Y)$. Then

$$\mathscr{O}_{\mathbb{A}^2}(U) = k[X^{\pm 1}, Y] \cap k[X, Y^{\pm 1}] = k[X, Y]$$

where the intersection happens in the **quotient field** $k(X, Y) = \mathrm{Quot}(k[X, Y])$ (also called the **fraction field**).

The above procedure is in fact general: If A is an integral domain, then

$$\mathscr{O}_{\mathrm{Sp}(A)}(U) = \bigcap_{D(f) \subseteq U} A[f^{-1}], \quad \text{intersect in } \mathrm{Quot}(A).$$

It suffices to take the intersection over an open covering $U = D(f_1) \cup \ldots \cup D(f_m)$ in fact.

4th Step: Morphisms. We had three steps to define the object $\mathrm{Sp}(A)$, but we also need to define morphisms between affine varieties. Let $X = \mathrm{Sp}(A)$ and $Y = \mathrm{Sp}(B)$ in the following.

Definition 1.17. A morphism from Y to X is a map $\varphi : Y \to X$ that preserves regular functions. That is, if $V \subseteq Y$ and $U \subseteq X$ are open with $\varphi(V) \subseteq U$, then for every $f \in \mathscr{O}_X(U)$, the composition

$$f \circ \varphi : V \longrightarrow k$$

lies in $\mathscr{O}_Y(V)$.

- In particular, if $\varphi : Y \to X$ is a morphism, we may apply this definition to $V = Y$ and $X = U$ and obtain a ring map

$$\begin{aligned} \varphi^* : A = \mathscr{O}_X(X) &\longrightarrow \mathscr{O}_Y(Y) = B \\ f &\longmapsto f \circ \varphi. \end{aligned}$$

This is the *pullback map* on the coordinate rings of X and Y.

- Conversely, any k-algebra homomorphism $\varphi^* : A \longrightarrow B$ k induces a morphism of affine varieties

$$\begin{aligned} \varphi = \mathrm{Sp}(\varphi^*) : Y &\longrightarrow X \\ B \supseteq \mathfrak{m}_B &\longmapsto (\varphi^*)^{-1}(\mathfrak{m}_B) \subseteq A. \end{aligned}$$

Theorem 1.18 (Hilbert's Nullstellensatz)**.** *The above functors define an equivalence of categories*

$$\begin{aligned} \left\{ \begin{matrix} \textit{reduced finite type} \\ k\textit{-algebras} \end{matrix} \right\}^{\mathrm{op}} &\xrightarrow{\cong} \left\{ \begin{matrix} \textit{affine varieties} \\ \textit{over } k \end{matrix} \right\} \\ A &\longmapsto \mathrm{Sp}(A), \\ \mathscr{O}_X(X) &\longleftarrow (X, \mathscr{O}_X). \end{aligned}$$

The general principle for passing between maps of varieties and algebra homomorphisms is the following. A morphism

$$k^n \xrightarrow{\varphi} k^m$$

given by an m-tuple of polynomials in n variables

$$(x_1, ..., x_n) \longmapsto (\varphi_1(x_1, ..., x_n), ..., \varphi_m(x_1, ..., x_n))$$

corresponds to the ring map

$$\begin{aligned} k[X_1, ..., X_n] &\xrightarrow{\widetilde{\varphi}} k[T_1, ..., T_m] \\ \varphi_i(X_1, ..., X_m) &\longleftarrow T_i. \end{aligned}$$

1.3. **Linear algebraic groups.** Reference: [1, §7.1, §7.2]

Throughout, k is an algebraically closed field of characteristic 0. We often abbreviate "linear algebraic group" by LAG.

Definition 1.19. An LAG over k is an affine variety G over k together with a group structure

$$m : G \times G \longrightarrow G$$

which is also a morphism of varieties.

Explanation. Let $X = \mathrm{Sp}(A)$, $Y = \mathrm{Sp}(B)$ be affine varieties over k. Then $X \times Y$ is the affine k-variety $\mathrm{Sp}(A \otimes_k B)$.

Why does this make sense?

- Mathematically:

$$\begin{aligned} A = k[T_1, \dots, T_n]/(f_1, \dots, f_m) &\longleftrightarrow X = V(f_1, \dots, f_m) \subseteq k^n, \\ B = k[S_1, \dots, S_r]/(g_1, \dots, g_s) &\longleftrightarrow Y = V(g_1, \dots, g_s) \subseteq k^r. \end{aligned}$$

$$\Longrightarrow\ A \otimes_k B = k[T_1, \dots, T_n, S_1, \dots, S_r]/(f_1, \dots, f_m, g_1, \dots, g_s)$$

and

$$\mathrm{Sp}(A \otimes_k B) = V(f_1, \dots, f_m) \times V(g_1, \dots, g_s) \subseteq k^{n+r},$$

just as desired for a product!

- $\mathrm{Sp}(A \otimes_k B)$ also has the universal property: For $Z = \mathrm{Sp}(C)$, we have

$$\begin{aligned}\mathrm{Mor}(Z, \mathrm{Sp}(A \otimes_k B)) &\underset{\text{Hilbert}}{=} \mathrm{Hom}_{k\text{-alg}}(A \otimes_k B, C)\\ &\underset{-\otimes_k -}{=} \mathrm{Hom}_{k\text{-alg}}(A, C) \times \mathrm{Hom}_{k\text{-alg}}(B, C)\\ &\underset{\text{Hilbert}}{=} \mathrm{Mor}(Z, X) \times \mathrm{Mor}(Z, Y).\end{aligned}$$

$\Longrightarrow$ $\mathrm{Sp}(A \otimes_k B)$ has univ. prop of $X \times Y$!

Conclusion. If $G = \mathrm{Sp}(A)$, then $G \times G = \mathrm{Sp}(A \otimes_k A)$. Condition on m is that it is defined by polynomials, equivalently that it comes from a map $A \longrightarrow A \otimes_k A$, and that it gives a group structure to the set underlying G.

Example 1.20. The *additive group* $\mathbb{G}_a$.

$\mathbb{G}_a = (k, +)$ with variety structure $k = \mathrm{Sp}(k[t])$.

$$+ : k \times k \longrightarrow k, \qquad (x, y) \longmapsto x + y$$

is the polynomial map corresponding on rings to

$$\begin{aligned} k[x] \otimes k[y] &\overset{\varphi}{\longleftarrow} k[t]\\ x \otimes 1 + 1 \otimes y &\longleftarrow t.\end{aligned}$$

Example 1.21. The *multiplicative group* $\mathbb{G}_m$.

$\mathbb{G}_m = (k^\times, *)$ with variety structure $k^\times = \mathrm{Sp}(k[t, t^{-1}])$.

$$* : k^\times \times k^\times \longrightarrow k^\times, \qquad (x, y) \longmapsto x \cdot y$$

induced from

$$\begin{aligned} k[x^{\pm 1}] \otimes k[y^{\pm 1}] &\longleftarrow k[t^{\pm 1}]\\ x \otimes y &\longleftarrow t.\end{aligned}$$

Remark 1.22. The above considered $k^\times$ as an open subset of $\mathbb{A}^1$. One can also treat a localization as a closed subset in an affine space of one dimension higher. For $\mathbb{G}_m$, this means identifying

$$\begin{aligned} k[t_1, t_2]/(t_1 t_2 - 1) &\xrightarrow{\sim} k[t, t^{-1}]\\ t_1, t_2 &\longmapsto t, t^{-1}.\end{aligned}$$

In these coordinates, multiplication is obviously given by polynomials:

$$\begin{array}{ccc} k^\times \times k^\times & \xrightarrow{m := *} & k^\times \\ \big\downarrow & & \big\downarrow \\ k^2 \times k^2 & \xrightarrow{\widetilde{m}} & k^2 \end{array} \qquad \begin{array}{ccc} (x, y) & \longmapsto & xy \\ \downarrow & & \downarrow \\ (x, x^{-1}, y, y^{-1}) & \longmapsto & (xy, (xy)^{-1}) \end{array}$$

$$\widetilde{m} : (x_1, x_2, y_1, y_2) \longmapsto (x_1 x_2, y_1 y_2).$$

Example 1.23. The *special linear* group SL_n.

$SL_n = (SL_n(k), *)$ with variety structure

$$SL_n = \mathrm{Sp}\big(k[T_{11}, \dots, T_{nn}]/(\det((T_{ij})_{i,j}) - 1)\big)$$

Here, $\det((T_{ij})_{i,j})$ is the determinant of the matrix with $T_{i,j}$ in the (i,j)-th entry. This is a polynomial in $k[T_{11},\dots,T_{nn}]$.

$$\begin{tikzcd} SL_n \times SL_n \arrow[r, dashed, "m := *"] \arrow[d, hook] & SL_n \arrow[d, hook, "\det = 1"] \\ k^{n^2} \times k^{n^2} \arrow[r, "\widetilde{m}"] & k^{n^2} \end{tikzcd}$$

$$\widetilde{m} : ((x_{ij}),(y_{ij})) \longmapsto \left(\sum_{k=1}^{n} x_{ik}y_{kj}\right)_{i,j}$$

The morphism $\widetilde{m}$ preserves the determinant 1 condition and hence restricts to a multiplication morphism on SL_n.

Example 1.24. The *general linear group* GL_n

$GL_n = (GL_n(k), *)$ with variety structure

$$GL_n = \mathrm{Sp}\big(k[T_{ij}, D]/(D \cdot \det(T_{ij}) - 1)\big) \subseteq k^{n^2} \times k.$$

The relation $D \cdot \det((T_{i,j})_{i,j}) - 1$ forces that D is the inverse of $\det((T_{ij})_{i,j})$, making the determinant invertible. This is the same as localizing at $\det((T_{i,j})_{i,j})$ as in Remark 1.22 before. We have

$$\begin{tikzcd} GL_n \times GL_n \arrow[r] \arrow[d, hook] & GL_n \arrow[d, hook] & (x_{ij}) \arrow[d, mapsto] \\ (k^{n^2} \times k) \times (k^{n^2} \times k) \arrow[r] & k^{n^2} \times k & ((x_{ij}), D_x) \end{tikzcd}$$

$$((x_{ij}), \det(x_{ij})^{-1}, (y_{ij}), D_y) \longmapsto ((x_{ij}) \cdot (y_{jk}), D_x \cdot D_y).$$

Example 1.25. Let $H \in \mathrm{M}_n(k)$ be a non-degenerate symmetric matrix, i.e. $H^t = H$ and $\det(H) \neq 0$. The *special orthogonal group* defined by H is

$$SO(H) = \{g \in SL_n(k) \mid g^t H g = H\} \subseteq SL_n.$$

Writing out the condition $g^t H g = H$ gives n^2 many quadratic polynomial equations $f_1, \dots, f_{n^2}$ in the g_{ij}. Correspondingly, the variety structure on $\mathrm{SO}(H)$ is by realizing it as

$$SO(H) = \mathrm{Sp}\big(k[g_{11},\dots,g_{nn}]/(f_1,\dots,f_{n^2}, \det(g_{ij}) - 1)\big).$$

The group structure is matrix multiplication from SL_n.

Observation. We always assume that k is algebraically closed, so there exists a unique non-degenerate, n variable quadratic form up to isometry.

$\implies$ After change of basis:

$$H = \begin{pmatrix} & \mathbb{1}_m \\ \mathbb{1}_m & \end{pmatrix} \quad n = 2m \quad \text{or} \quad H = \begin{pmatrix} 1 & & \\ & & \mathbb{1}_m \\ & \mathbb{1}_m & \end{pmatrix} \quad n = 2m+1.$$

Here $\mathbb{1}_m$ denotes the $m \times m$ identity matrix. For comparison, recall that over $\mathbb{R}$ a quadratic form is classified by its signature. Correspondingly, there would be several n variable orthogonal groups.

Example 1.26. The *symplectic group* Sp_{2m} over k is defined by

$$Sp_{2m}(k) = \left\{ g \in M_{2m}(k) \;\middle|\; g^t \begin{pmatrix} & \mathbb{1}_m \\ -\mathbb{1}_m & \end{pmatrix} g = \begin{pmatrix} & \mathbb{1}_m \\ -\mathbb{1}_m & \end{pmatrix} \right\} \subseteq k^{4m^2}.$$

It is described by $4m^2$ polynomial equations with matrix multiplication just as the special orthogonal group before.

Remark 1.27. SL_n, SO_{2m}, SO_{2m+1} and Sp_{2m} are called the *classical groups.*

References

[1] Humphreys, James E.; *Linear algebraic groups*, Graduate Texts in Mathematics **21**, Springer-Verlag New York-Heidelberg,1975.