Cyclic Base Change for $\mathbf{GL}_n$

Program suggestion: Andreas Mihatsch, Fabio La Rosa

Overview

In this seminar, we study Labesse's [6] proof of the existence of cyclic base change for automorphic representations of GL_n. It is based on a comparison of trace formulae and on the fundamental lemma for cyclic base change. Let us briefly describe what these are and how the come up during our seminar.

Let G be an anisotropic reductive group over $\mathbb{Q}$. For example, G could be a unitary group such that $G(\mathbb{R})$ is compact, or G could be the elements of norm 1 in a central division algebra $D/\mathbb{Q}$. Then $G(\mathbb{Q})\backslash G(\mathbb{A})$ is compact and $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is a Hilbert space direct sum with finite multiplicities m_π of all irreducible cuspidal automorphic representations of G,

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \widehat{\bigoplus_{\pi \text{ irred. cusp.}}} \pi^{\oplus m_\pi}.$$

For every $f \in C_c^\infty(G(\mathbb{A}))$, the action of f on this L^2-space is of trace class, and the trace formula expresses its trace in two different ways:

$$\sum_{\pi \text{ irred. cusp.}} m_\pi \cdot \mathrm{tr}(f \mid \pi) \;=\; \sum_{\overline{\gamma} \in G(\mathbb{Q})\backslash\backslash G(\mathbb{Q})} J(\gamma, f). \tag{1}$$

The left hand side is called the spectral side. The right hand side involves the orbital integrals $J(\gamma, f)$ for all conjugacy classes $\overline{\gamma} \in G(\mathbb{Q})$ and is called the geometric side.

For isotropic G such as $G = \mathrm{GL}_n$, the action of f on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$ is no longer of trace class. Still, Arthur has defined distributions that are very similar to the traces and orbital integrals that come up in (1), and has proved a trace formula for them. The aim of the first two talks in our seminar is to learn about his definitions when $G = \mathrm{GL}_n$.

Let E/F be a cyclic extension of number fields. Cyclic base change is about transporting irreducible automorphic representations from $\mathrm{GL}_{n,F}$ to $\mathrm{GL}_{n,E}$. After the first two talks, we have trace formulas for both $\mathrm{GL}_{n,F}$ and $\mathrm{GL}_{n,E}$, and our aim is to compare them. Loosley speaking, we want to show that certain multiplicities m_π are non-zero which will be equivalent to the existence of the base change.

The basic idea of this comparison is to construct many pairs of functions (f_F, f_E) for which the geometric sides of the two trace formulas become equal. This is based on the norm map from conjugacy classes of $\mathrm{GL}_{n,E}$ to those of $\mathrm{GL}_{n,E}$. In particular, we will prove the base change fundamental lemma for GL_n which states an explicit matching of orbital integrals for elements of spherical Hecke algebras.

Finally, we can reap the rewards: We will prove the existence of base change both globally and locally, and also see an application to inner forms of GL_n.

References in the following are for [6]. Base change always means cyclic base change.

Helpful hint: We only need the case $\widetilde{L} = \mathrm{GL}_{n,E} \rtimes \langle\theta\rangle$ of Labesse's notation, so feel free to specialize all notation to this case for concreteness. The case $E = F$ recovers the usual GL_n.

1 Trace Formula

1. Geometric Distributions (I.1-3 and II.1)

The objects appearing in the geometric side of the trace formula are orbital integrals: distributions attached to rational conjugacy classes. Begin by explaining the geometric side in the case of a compact quotient as in [3, Lecture I.2]. (This is precisely the RHS of (1) above.) Use the rest of your talk to define the geometric side of Arthur's trace formula for our setting. More precisely, we need the definitions on the right hand side of II.1.2 for $\mathrm{GL}_{n,E} \rtimes \langle\theta\rangle$.

Remark: An explicit description of weights for GL_2 and GL_3 may be found in [3, Lecture IV, Section 3].

2. Spectral Distributions (I.4 and II.1-2)

The trace formula we need is given by the equality of the geometric expansion in Proposition II.1.2 and of the spectral expansion in Proposition II.2.1. The objects appearing in the spectral expansion are characters of representations parabolically induced from cuspidal representations of Levi subgroups. Review the notion of cuspidal automorphic representation and the decomposition of the L^2-spectrum ([4], Section 9.2 and 10.4) to define the terms appearing in Proposition II.2.1. State the trace formula in our setting.

2 Fundamental Lemma for Base Change

3. Norm Map and Endoscopic Transfer (III.1-4)

Introduce the norm map (III.1), the transfer of characters (III.2), and the transfer of test functions (III.3). Present Proposition III.4.1 and Corollary III.4.2 which state that transfer of characters can be detected by transfer of test functions. The proof of Corollary III.4.2 uses Weyl's integration formula, which can be found in [5], Theorem 8.64.

4. Elementary Functions (IV.1-3)

Elementary functions are objects of non-Archimedean harmonic analysis: they enjoy several properties which allow to compute spectral and geometric distributions of spherical Hecke operators in terms of elementary functions. Present the main results of IV.1-3, which are Proposition IV.2.2 and Proposition IV.3.6.

5. and 6. Fundamental Lemma (III.5, IV.5, V.1 and V.3-6)

The aim of the talk is to prove the fundamental lemma (Proposition V.6.3). It is established by a local-global approach in which it is assumed inductively that the fundamental lemma holds for all proper Levi subgroups. The proof also uses the results from III.5 on strong association at split (Archimedean and non-Archimedean places) in combination with the theory of elementary functions.

3 Cyclic Base Change and Applications

7. Global Base Change (VI.1-4)

Prove the existence of cyclic base change (Theorem VI.4.1), the main result of our seminar. The key intermediate result is the comparison of trace formulae (Proposition VI.1.3). You may simply state without proof the results from Sections VI.2 and Sections VI.3 when needed.

8. Local Cyclic Base Change and Inner Forms (VI.6)

Present the two further applications that are contained in Labesse's article. First, the existence of local base change (Theorem VI.5.1). Its proof uses the existence of global base change. It relies on the Langlands classification and on an ingenious reduction step: every discrete representation is the local component of a cupidal automorphic representation ([2]). Second, Proposition VI.6.1 which, under some assumptions, yields the existence of base change for inner forms of GL_n.

References

[1] J. Arthur, L. Clozel. *Simple algebras, base change, and the advanced theory of the trace formula*, Princeton University Press, 1989.

[2] L. Clozel. *On limit multiplicities of discrete series representations in spaces of automorphic forms*, Invent. Math. 83(2), 265-284, 1986.

[3] S. Gelbart. *Lectures on the Arthur-Selberg trace formula*, A.M.S., University Lecture Series, vol. 9, 1996.

[4] J. R. Getz. *An introduction to automorphic representations*, Springer, GTM vol. 300, 2024.

[5] A. W. Knapp. *Lie groups beyond an introduction*, Second Edition, Birkhauser, 2002.

[6] J-P. Labesse. *Noninvariant base change identities*, Mémoires de la S.M.F. 2^e série, tome 61, 1995.