GAUSSIAN TEST FUNCTIONS FOR THE JACQUET–RALLIS RELATIVE TRACE FORMULA

Contents

1. Introduction

The relative trace formula comparison of Jacquet–Rallis [12], which lead to a proof of the unitary Gan–Gross–Prasad Conjecture [5, 4], relies on a comparison of orbital integrals between a GL_n-setting and a unitary setting. The main local problems in this context are the existence of smooth transfer and the fundamental lemma.

Over non-archimedean fields, the existence of transfer was proved by Wei Zhang [25]. The fundamental lemma has three independent proofs: one via equal characteristic methods by Zhiwei Yun and Gordon [11, 23], one via global theta series by Wei Zhang [26], and a local one based on Fourier transforms by Beuzart–Plessis [3].

Over $\mathbb{R}$, it was proved by Hang Xue [22] that a dense subspace of Schwartz functions is transferable which was sufficient for the global applications in [5, 4]. It is conjectured that transfer exists for all Schwartz functions.

1.1. **Gaussian test functions.** In arithmetic situations, one is often interested in specific test functions at the archimedean place. For example, when studying the cohomology of Shimura varieties, one considers so-called Lefschetz functions, cf. [15, §3]. Their orbital integrals are non-zero only for elliptic group elements which means they can be understood as coming by transfer from the compact inner form of the group in question.

In the trace formula setting of Jacquet–Rallis, the analogous kind of functions are those coming by transfer from the compact unitary group $U(n+1)$. For example, a transfer of the identity function is used in Wei Zhang's relative trace formula approach to the Arithmetic Gan–Gross–Prasad Conjecture [24]. Such transfers were named *Gaussian test functions* in [21], and these are the test functions from the title of the paper.

The purpose of our paper is to give a simple, direct, and local construction of Gaussian test functions. Their existence was already known before from [5, Proposition 4.11]. Our explicit construction has the advantage that it also allows to study derivatives of orbital integrals. This matters, for example, during the proof of the arithmetic fundamental lemma [26, §12], [17, §10]; and also plays a role in ongoing work of the authors on arithmetic generating series.

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1.2. **Main result.** We now give a precise description of our main result and also introduce concepts that will be used throughout the paper. Consider the real manifold

$$S_{n+1} := \{\gamma \in \mathrm{GL}_{n+1}(\mathbb{C}) \mid \gamma\overline{\gamma} = 1\}. \tag{1.1}$$

The group $\mathrm{GL}_n(\mathbb{R})$ acts on it by conjugation via $g \mapsto \mathrm{diag}(g, 1)$. We denote by $[S_{n+1}]_{\mathrm{rs}}$ the set of regular semi-simple orbits.

In a similar way, for every hermitian $\mathbb{C}$-vector space V, the group $U(V)$ acts by conjugation on $U(V \oplus \mathbb{C})$. Here, $\mathbb{C}$ is viewed with the standard hermitian form of signature $(1, 0)$. We again denote by $[U(V \oplus \mathbb{C})]_{\mathrm{rs}}$ the set of regular semi-simple orbits. The orbit matching of Jacquet and Rallis [12] defines a bijection

$$[S_{n+1}]_{\mathrm{rs}} \xrightarrow{\sim} \coprod_{r+s=n} [U(V_{(r,s)} \oplus \mathbb{C})]_{\mathrm{rs}}, \tag{1.2}$$

where $V_{(r,s)}$ is a choice of hermitian $\mathbb{C}$-vector space of signature (r, s). Suppose $\phi \in \mathcal{S}(S_{n+1})$ is a Schwartz function and $\gamma \in S_{n+1}$ regular semi-simple. Jacquet and Rallis introduced the orbital integral

$$\mathrm{Orb}(\gamma, \phi) = \epsilon(\gamma) \int_{\mathrm{GL}_n(\mathbb{R})} \phi(g^{-1}\gamma g)\eta(g)\, dg \tag{1.3}$$

where η is the sign character $\eta(g) = \mathrm{sign}(\det(g))$, where dg is a fixed choice of Haar measure, and where $\epsilon(\gamma) \in \mathbb{C}^\times$ is the transfer factor defined in §5.1.

Definition 1.1. (1) A regular semi-simple element $\gamma \in S_{n+1}$ is said to *match to signature* (r, s) if its matching orbit under (1.2) lies in the signature (r, s) component.

(2) A Schwartz function $\phi \in \mathcal{S}(S_{n+1})$ is said to be a *Gaussian test function* if its orbital integrals are given by

$$\mathrm{Orb}(\gamma, \phi) = \begin{cases} 1 & \gamma \text{ matches to signature } (n, 0) \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Our main result is the following theorem.

Theorem 1.2. *Gaussian test functions in the sense of Definition 1.1 exist. Moreover, they can be explicitly constructed in terms of Kudla–Millson theory.*

In fact, since the groups $U(n)$ and $U(n + 1)$ are compact and connected, the matrix coefficients of their irreducible representations are algebraic functions. Theorem 1.2 then immediately extends to a description of transfer for such matrix coefficients (see Corollary 5.5).

We will deduce Theorem 1.2 from an analogous statement for Lie algebras, and the bulk of our paper is about this Lie algebra variant.

1.3. **Strategy of proof.** Consider the tangent space at the identity of S_{n+1}:

$$\mathfrak{s}_{n+1} = \{y \in M_{n+1}(\mathbb{C}) \mid y + \overline{y} = 0\}.$$

Also define $\mathfrak{u}(V) = \mathrm{Lie}(U(V))$. Then $\mathrm{GL}_n(\mathbb{R})$ acts by conjugation on $\mathfrak{s}_{n+1}$, and $U(V)$ acts by conjugation on $\mathfrak{u}(V \oplus \mathbb{C})$ as before. By the infinitesimal trace formula comparison [12, 8], there is again a bijective matching of regular semi-simple orbits

$$[\mathfrak{s}_{n+1}]_{\mathrm{rs}} \xrightarrow{\sim} \coprod_{r+s=n} [\mathfrak{u}(V_{(r,s)} \oplus \mathbb{C})]_{\mathrm{rs}}. \tag{1.5}$$

The precise definition will be recalled in §2.1. We again say that $y \in \mathfrak{s}_{n+1,\mathrm{rs}}$ *matches to signature* (r, s) if its matching orbit under (1.5) lies in the (r, s)-component. For $y \in \mathfrak{s}_{n+1,\mathrm{rs}}$ and $\Phi \in \mathcal{S}(\mathfrak{s}_{n+1})$, there is again an orbital integral (Definition 2.5)

$$\mathrm{Orb}(y, \Phi) = \varepsilon(y) \int_{\mathrm{GL}_n(\mathbb{R})} \Phi(g^{-1}yg)\eta(g)\, dg. \tag{1.6}$$

Both $\mathfrak{s}_{n+1}$ and each $\mathfrak{u}(V_{(r,s)} \oplus \mathbb{C})$ are quadratic spaces with quadratic form $Q(z) = -\mathrm{tr}(z^2)$, and these forms are constant along orbits. If two elements $y \in \mathfrak{s}_{n+1,\mathrm{rs}}$ and $x \in \mathfrak{u}(V_{(r,s)} \oplus \mathbb{C})_{\mathrm{rs}}$ match under (1.5), then $Q(x) = Q(y)$. The quadratic form on $\mathfrak{u}(V_{(n,0)} \oplus \mathbb{C})$ is positive definite, so we can consider the Gaussian $\Psi \in \mathcal{S}(\mathfrak{u}(V_{(n,0)} \oplus \mathbb{C}))$ defined by

$$\Psi(x) = e^{-2\pi Q(x)}.$$

The following definition is the infinitesimal analogue of Definition 1.1.

Definition 1.3. A *Gaussian test function* on $\mathfrak{s}_{n+1}$ is a Schwartz function $\Phi \in \mathcal{S}(\mathfrak{s}_{n+1})$ that is a smooth transfer of Ψ. That is, for all regular semi-simple $y \in \mathfrak{s}_{n+1}$,

$$\mathrm{Orb}(y, \Phi) = \begin{cases} e^{-2\pi Q(y)} & y \text{ matches to signature } (n, 0) \\ 0 & \text{otherwise.} \end{cases}$$

Having passed to this Lie algebra setting, our main idea is to apply Kudla–Millson theory [13, 16], in particular their construction of certain differential forms on symmetric spaces attached to orthogonal groups. Let $X = \mathrm{GL}_n(\mathbb{R})/O(n)$ denote the symmetric space attached to GL_n, and let D be the symmetric space attached to $SO(\mathfrak{s}_{n+1})$. The action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathfrak{s}_{n+1}$ is an orthogonal representation,

$$\rho\colon\ \mathrm{GL}_n(\mathbb{R}) \longrightarrow SO(\mathfrak{s}_{n+1}).$$

It descends to a closed immersion

$$\alpha\colon X \longrightarrow D$$

at the level of symmetric spaces, cf. Section 3.1. Identifying $T_e(X) = \mathrm{Sym}_n(\mathbb{R})$, the pullback $\alpha^*(\varphi_{\mathrm{KM}})$ of the Kudla–Millson form φ_{KM} lies in $\mathcal{S}(\mathfrak{s}_{n+1}) \otimes \det(\mathrm{Sym}_n(\mathbb{R}))$. Let $\omega \in \det(\mathrm{Sym}_n(\mathbb{R}))$ be the properly oriented generator that defines the chosen Haar measure on $\mathrm{GL}_n(\mathbb{R})$ under the Iwasawa decomposition, see Section 4.4 for details.

Theorem 1.4. *Let $\Phi \in \mathcal{S}(\mathfrak{s}_{n+1})$ be the Schwartz function characterized by the identity*

$$\alpha^*(\varphi_{\mathrm{KM}}) = \Phi \otimes \omega. \tag{1.7}$$

Then, up to a constant multiple, Φ is a Gaussian test function in the sense of Definition 1.3.

The reader is referred to Theorem 4.15 for explicit normalizations.

For the proof of Theorem 1.4, we analyse the intersection behaviour of $\alpha(X)$ with the Kudla–Millson cycles D_y, $y \in \mathfrak{s}_{n+1,\mathrm{rs}}$. On the one hand, taking into account orientations, this intersection number is $\varepsilon(y)$ or 0, depending on whether y matches to signature $(n, 0)$ or not (Proposition 3.2).

On the other hand, the Kudla–Millson form is, in a certain sense, dual to the cycles D_y. Once the corresponding convergence statement is established, its integrals over X are hence equal to the above intersection numbers, which ultimately completes the proof.

Two points of this construction seem miraculous to us. The first is the numerical coincidence that underlies the definition of Φ via (1.7): the negative part of the signature of $\mathfrak{s}_{n+1}$ agrees with the dimension of the symmetric space X.

The second is the signs that are hidden in the construction. Namely, the differential forms of Kudla–Millson transform with a certain quadratic character. Specialized to our situation, we obtain that $\alpha^*(\varphi_{\mathrm{KM}})$ is $(O(n),\eta^n)$-invariant. At the same time, the character of $O(n)$ acting on $\det(\mathrm{Sym}_n(\mathbb{R}))$ is η^{n-1}. Taken together, this implies that Φ is $(O(n),\eta)$-invariant which aligns with the character in Jacquet–Rallis's orbital integrals (1.6).

Finally, we remark that the Kudla–Millson form φ_{KM} and, by extension, the function Φ in (1.7) are completely explicit. We refer to Examples 4.16 and 4.17 for the cases $n=1$ and $n=2$.

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2. Setting

We give precise definitions of orbit matching and of the orbital integral in Definition 1.3.

2.1. **Orbits and matching.** A reference for the following statements is [8, §2.1 and §2.2]. Set $G=\mathrm{GL}_n(\mathbb{R})$. An element $y\in\mathfrak{s}_{n+1}$ is *regular semi-simple* if the stabilizer G_y is trivial, and the orbit $G\cdot y\subset\mathfrak{s}_{n+1}$ Zariski closed. A concrete characterization of this property is as follows. Write y in a block matrix form

$$y=\begin{pmatrix} y_0 & v\\ w & d\end{pmatrix}\ \in\ \begin{pmatrix}\mathfrak{s}_n & i\cdot\mathbb{R}^n\\ i\cdot\mathbb{R}_n & i\cdot\mathbb{R}\end{pmatrix}. \tag{2.1}$$

Here and in the following, we use $\mathbb{R}^n$ and $\mathbb{R}_n$, as well as $\mathbb{C}^n$ and $\mathbb{C}_n$, to respectively denote column and row vectors. Then

$$y \text{ regular semi-simple} \quad\Longleftrightarrow\quad \mathbb{C}[y_0]\cdot v=\mathbb{C}^n \text{ and } w\cdot\mathbb{C}[y_0]=\mathbb{C}_n. \tag{2.2}$$

The invariant of y is defined as the tuple

$$\mathrm{Inv}(y):=\big(\mathrm{char}(y_0;T),\ wv,\ wy_0v,\ \ldots,\ wy_0^{n-1}v,\ d\big). \tag{2.3}$$

It can be understood as the $\mathbb{R}$-point of the GIT quotient $G\,\backslash\!\backslash\,\mathfrak{s}_{n+1}\cong\mathbb{A}^{2n+1}$ defined by y. It is known, see [8, Lemma 2.1.5.1], that two regular semi-simple elements y and y' satisfy

$$G\cdot y=G\cdot y' \quad\Longleftrightarrow\quad \mathrm{Inv}(y)=\mathrm{Inv}(y'). \tag{2.4}$$

We turn to the unitary side. Let V be an n-dimensional hermitian $\mathbb{C}$-vector space. An element $x\in\mathfrak{u}(V\oplus\mathbb{C})$ is *regular semi-simple* if its stabilizer $U(V)_x$ is trivial and if the orbit $U(V)\cdot x\subseteq\mathfrak{u}(V\oplus\mathbb{C})$ is Zariski closed. Write x in block matrix form

$$x=\begin{pmatrix} x_0 & u\\ -u^* & d\end{pmatrix}\ \in\ \begin{pmatrix}\mathfrak{u}(V) & V\\ V^* & i\cdot\mathbb{R}\end{pmatrix} \tag{2.5}$$

where we identified $\mathrm{Hom}(\mathbb{C},V)\xrightarrow{\sim}V$ in the obvious way. Similarly to before, x is regular semi-simple if and only if $\mathbb{C}[x_0]\cdot u=V$. The invariant of x is defined as

$$\mathrm{Inv}(x):=\big(\mathrm{char}(x_0;T),\ -(u,u),\ -(u,x_0u),\ \ldots,\ -(u,x_0^{n-1}u),\ d\big). \tag{2.6}$$

As before, two regular semi-simple elements x and x' lie in the same $U(V)$-orbit if and only if $\mathrm{Inv}(x)=\mathrm{Inv}(x')$.

Definition 2.1. We denote by $\mathfrak{s}_{n+1,\mathrm{rs}}$ and $\mathfrak{u}(V \oplus \mathbb{C})_{\mathrm{rs}}$ the subsets of regular semi-simple elements. Two elements $y \in \mathfrak{s}_{n+1,\mathrm{rs}}$ and $x \in \mathfrak{u}(V \oplus \mathbb{C})_{\mathrm{rs}}$ are said to *match* if $\mathrm{Inv}(y) = \mathrm{Inv}(x)$.

Recall that $V_{(r,s)}$ is our notation for a hermitian $\mathbb{C}$-vector space of signature (r,s). For concreteness, we choose $V_{(r,s)} = \mathbb{C}^n$ with standard hermitian form $\mathrm{diag}(1_r, -1_s)$ in the following. We have already stated in (1.5) that matching defines a bijection between regular semi-simple orbits on the general linear and unitary side, see [8, Propositions 2.1.5.2 and 2.2.4.1].

Lemma 2.2. *A regular semi-simple element $y \in \mathfrak{s}_{n+1}$ matches to signature $(n,0)$ if and only if y has an orbit representative of the form*

$$i \cdot \begin{pmatrix} \lambda_1 & & & \mu_1 \\ & \ddots & & \vdots \\ & & \lambda_n & \mu_n \\ \mu_1 & \cdots & \mu_n & d \end{pmatrix}. \tag{2.7}$$

We remark that an element of the form (2.7) is regular semi-simple if and only if the λ_j are pairwise different and the μ_j all non-zero. This is clear from (2.2).

Proof. Assume that $y \in \mathfrak{s}_{n+1}$ and $x \in \mathfrak{u}(V_{(n,0)} \oplus \mathbb{C})$ are regular semi-simple and matching. We use the notation of (2.1) and (2.5). Since x_0 lies in $\mathfrak{u}(V_{(n,0)})$, since $V_{(n,0)}$ is definite, and since x is regular semi-simple, the characteristic polynomial of x_0 is separable with eigenvalues in $i \cdot \mathbb{R}$. By the definition of matching, $\mathrm{char}(y_0; T) = \mathrm{char}(x_0; T)$. Thus y_0 is G-diagonalizable which means that y is G-conjugate to an element of the form

$$y' = i \cdot \begin{pmatrix} \lambda_1 & & & \mu_1 \\ & \ddots & & \vdots \\ & & \lambda_n & \mu_n \\ \mu_1' & \cdots & \mu_n' & d \end{pmatrix}. \tag{2.8}$$

None of the μ_k and μ_k' vanishes by (2.2). The $\mathbb{C}$-algebras $\mathbb{C}[y_0']$ and $\mathbb{C}[x_0]$ are isomorphic via $y_0' \mapsto x_0$. For $k = 1, \ldots, n$, let $\pi_k(y_0') \in \mathbb{C}[y_0']$ denote the idempotent for the λ_k-eigenspace. Then

$$\begin{aligned} i\mu_k' \cdot i\mu_k &= (i\mu_1', \ldots, i\mu_n') \cdot \pi_k(y_0') \cdot {}^t(i\mu_1, \ldots, i\mu_n) \\ &= -(u, \pi_k(x_0)u) < 0, \end{aligned}$$

where the second equality comes from the definition of matching, and where the inequality comes from the definiteness of $V_{(n,0)}$. So

$$\mu_k \cdot \mu_k' = (u, \pi_k(x_0)u) > 0.$$

Conjugating (2.8) by the diagonal matrix $\mathrm{diag}(|\mu_1'/\mu_1|^{1/2}, \ldots, |\mu_n'/\mu_n|^{1/2})$ brings it into the form of (2.7).

Conversely, assume that y has the form (2.7). Set $x = y$ and view it as an element of $M_{n+1}(\mathbb{C})$. Since x satisfies ${}^t\overline{x} = -x$, it lies in the Lie algebra $\mathfrak{u}(V_{(n,0)} \oplus \mathbb{C})$. The definition of $\mathrm{Inv}(x)$ does not depend on whether we view it as element of $\mathfrak{s}_{n+1}$ or of $\mathfrak{u}(V_{(n,0)} \oplus \mathbb{C})$. So this shows that y matches to signature $(n,0)$ as claimed. $\square$

Lemma 2.3. *Consider the open subset*

$$\mathfrak{S} = \{y \in \mathfrak{s}_{n+1,\mathrm{rs}} \mid y \text{ matches to signature } (n,0)\}. \tag{2.9}$$

Then $\mathfrak{S}$ has two connected components which are interchanged by any $g \in G$ with $\det(g) < 0$.

Proof. By Lemma 2.2, every element of $\mathfrak{S}$ has a representative of the form (2.7). Acting with permutation matrices from G, and with diagonal matrices of the form $\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, $\varepsilon_k \in \{\pm 1\}$, we may even find such a representative with $\lambda_1 > \ldots > \lambda_n$ and all $\mu_k > 0$. Thus, if we let $\mathfrak{R} \subseteq \mathfrak{s}_{n+1}$ denote the set of all matrices of the form (2.7) that satisfy these conditions, then

$$\mathfrak{R} \subset \mathfrak{S} \quad \text{and} \quad \mathfrak{S} = G \cdot \mathfrak{R}.$$

The manifold $\mathfrak{R}$ is connected, the group G has two connected components, and regular semi-simple elements have trivial stabilizer. It follows that $\mathfrak{S}$ has two connected components which are interchanged by elements with negative determinant as claimed. □

2.2. **Orbital integrals.** Recall that $\eta\colon G \to \{\pm 1\}$ denotes the sign character. The transfer factors used in [12, 5] are adapted to a global trace formula setting which we do not need here, so we use the following simple definition:

Definition 2.4. A *transfer factor* is a locally constant function

$$\varepsilon\colon \mathfrak{s}_{n+1,\mathrm{rs}} \longrightarrow \{\pm 1\},$$

that satisfies $\varepsilon(g^{-1}yg) = \eta(g)\varepsilon(y)$ for all $g \in G$ and $y \in \mathfrak{s}_{n+1,\mathrm{rs}}$.

Note that the orbital integrals of Gaussian test functions are non-zero only for $y \in \mathfrak{S}$, and $\varepsilon|_{\mathfrak{S}}$ is unique up to sign by Lemma 2.3. So for the purposes of our article, choosing ε can be understood as fixing one of the two (G,η)-invariant sign functions on $\mathfrak{S}$.

We fix a transfer factor ε for the rest of the article. Recall that we also already fixed a Haar measure on G in §1.

Definition 2.5. Let $\Phi \in \mathcal{S}(\mathfrak{s}_{n+1})$ be a Schwartz function and $y \in \mathfrak{s}_{n+1}$ a regular semi-simple element. The orbital integral $\mathrm{Orb}(y,\Phi)$ is defined by

$$\mathrm{Orb}(y,\Phi) := \varepsilon(y) \int_G \Phi(g^{-1}yg)\eta(g)\, dg. \tag{2.10}$$

It only depends on the orbit $G \cdot y$.

3. Intersection numbers

3.1. **Symmetric spaces.** Let $G^0 = \mathrm{GL}_n(\mathbb{R})^{\det > 0}$ be the identity connected component. Let $K = SO(n) \subset G^0$ be the standard maximal compact subgroup and let $X = G^0/K$ be the corresponding symmetric space.

We denote by Sym_n and Skew_n the real vector spaces of symmetric (resp. skew-symmetric) $(n \times n)$-matrices. We write $\mathrm{Sym}_n^{>0}$ for the positive definite symmetric matrices. We can describe X as

$$X \xrightarrow{\sim} \mathrm{Sym}_n^{>0}, \quad gK \longmapsto {}^t g^{-1} \cdot g^{-1}. \tag{3.1}$$

Recall that we endowed $\mathfrak{s}_{n+1}$ with the quadratic form $Q(y) = -\mathrm{tr}(y^2)$. There is an orthogonal decomposition

$$\mathfrak{s}_{n+1} = (i \cdot \mathrm{Sym}_{n+1}) \overset{\perp}{\oplus} (i \cdot \mathrm{Skew}_{n+1})$$

where the quadratic form is positive definite on the first, and negative definite on the second summand. Let

$$K_{SO} = SO(i \cdot \mathrm{Sym}_{n+1}) \times SO(i \cdot \mathrm{Skew}_{n+1})$$

be the corresponding maximal compact subgroup of the identity connected component $SO(\mathfrak{s}_{n+1})^0$, and let

$$D := SO(\mathfrak{s}_{n+1})^0 \,/\, K_{SO}$$

be the quotient symmetric space. A concrete description of D is given by

$$\begin{aligned} D &\xrightarrow{\sim} \{W \subseteq \mathfrak{s}_{n+1} \mid Q|_W < 0,\ \dim(W) = n(n+1)/2\} \\ hK_{SO} &\longmapsto h \cdot (i \cdot \mathrm{Skew}_{n+1}). \end{aligned} \tag{3.2}$$

Recall that $\rho\colon G \longrightarrow SO(\mathfrak{s}_{n+1})$ denotes the orthogonal representation that defines the action of G on $\mathfrak{s}_{n+1}$, i.e. for $g \in G$ and $y \in \mathfrak{s}_{n+1}$, we set

$$\rho(g)y = \begin{pmatrix} g & \\ & 1 \end{pmatrix} y \begin{pmatrix} g^{-1} & \\ & 1 \end{pmatrix}. \tag{3.3}$$

To simplify notation, we will often write $g \cdot y$ for $\rho(g)y$. The map ρ descends to a closed immersion

$$\alpha\colon X \longrightarrow D \tag{3.4}$$

of real manifolds, and our next aim is to describe it in terms of (3.1) and (3.2).

Suppose that $H \in M_n(\mathbb{R})$ is a symmetric $(n \times n)$-matrix with $\det(H) \neq 0$. Then we denote by $\mathrm{Sym}(H)$ and $\mathrm{Skew}(H)$ the real vector spaces of matrices that are (skew-)symmetric with respect to H. That is,

$$\begin{aligned} \mathrm{Sym}(H) &= \{A \in M_n(\mathbb{R}) \mid {}^tA = HAH^{-1}\} \\ \mathrm{Skew}(H) &= \{A \in M_n(\mathbb{R}) \mid {}^tA = -HAH^{-1}\}. \end{aligned}$$

Observe that for $g \in \mathrm{GL}_n(\mathbb{R})$,

$$g \cdot \mathrm{Sym}(H) \cdot g^{-1} = \mathrm{Sym}({}^tg^{-1} \cdot H \cdot g^{-1}), \tag{3.5}$$

and analogously for $\mathrm{Skew}(H)$. With this terminology in place, α is given by

$$\begin{aligned} \alpha\colon \mathrm{Sym}_n^{>0} &\longrightarrow D \\ H &\longmapsto i \cdot \mathrm{Skew}\left(\begin{pmatrix} H & \\ & 1 \end{pmatrix}\right). \end{aligned} \tag{3.6}$$

3.2. **Kudla–Millson cycles.** For a vector $y \in \mathfrak{s}_{n+1}$, Kudla–Millson introduced the totally geodesic submanifold

$$D_y = \{W \in D \mid y \perp W\}.$$

Clearly, as long as $y \neq 0$,

$$D_y = \begin{cases} \text{symmetric space for } SO(\langle y\rangle^\perp) & \text{if } Q(y) > 0 \\ \emptyset & \text{if } Q(y) \leq 0. \end{cases}$$

In particular, D_y is of codimension $n(n+1)/2$ if $Q(y) > 0$.

Proposition 3.1. *Let $y \in \mathfrak{s}_{n+1}$ be regular semi-simple. Then*

$$|X \cap D_y| = \begin{cases} 1 & \text{if } y \text{ matches to signature } (n,0) \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Moreover, in the first case, the intersection $X \cap D_y$ is transversal.

Proof. Assume that y matches to signature $(n,0)$. (This in particular implies that $Q(y) > 0$.) By the invariance property $g \cdot D_y = D_{g \cdot y}$, we may assume that y is of the diagonal form (2.7). In particular $y \in i \cdot \mathrm{Sym}_n$, which is equivalent to $y \perp i \cdot \mathrm{Skew}_{n+1}$. In terms of (3.6), this means that $H = 1_n$ lies in $X \cap D_y$, and hence $X \cap D_y \neq \emptyset$.

Conversely, assume that $X \cap D_y$ is non-empty. This means that there exists a positive definite quadratic form $H \in \mathrm{Sym}_n$ such that $y \perp i \cdot \mathrm{Skew}\left(\begin{pmatrix} H & \\ & 1 \end{pmatrix}\right)$, which is equivalent to $y \in$

$i\cdot\mathrm{Sym}\left(\begin{pmatrix} H & \\ & 1\end{pmatrix}\right)$. In terms of the block coordinates from (2.1), this implies y_0 is diagonalizable with eigenvalues in $i\cdot\mathbb{R}$. Put differently, the G-orbit $G\cdot y$ contains an element y' of the form (2.8):

$$y' = i\cdot\begin{pmatrix}\lambda_1 & & & \mu_1\\ & \ddots & & \vdots\\ & & \lambda_n & \mu_n\\ \mu_1' & \cdots & \mu_n' & d\end{pmatrix}. \tag{3.8}$$

Since y' is regular semi-simple, the eigenvalues $\lambda_1,\dots,\lambda_n$ are pairwise different. Moreover, since $X\cap D_y \xrightarrow{\sim} X\cap D_{y'}$, there exists a positive definite symmetric matrix H' such that $y'\in i\cdot\mathrm{Sym}\left(\begin{pmatrix} H' & \\ & 1\end{pmatrix}\right)$. Since y_0' is diagonal, the only possibility for H' is to be diagonal. Since H' is positive definite, its entries are strictly positive. In terms of (3.8), this implies that μ_k and μ_k' have the same sign, for all $k=1,\dots,n$. Hence y' is G-conjugate to an element of the form (2.7) and hence matches to signature $(n,0)$ by Lemma 2.2. Moreover, H' is uniquely determined by $H'=\mathrm{diag}(\mu_1'/\mu_1,\dots,\mu_n'/\mu_n)$, which proves the set-theoretic statement (3.7).

It is left to show the transversality of the intersection. Assume that there exists a non-zero tangent vector to a point $e\in X\cap D_y$,

$$0\neq v\in T_e(X)\cap T_e(D_y)$$

where the intersection is taken in the tangent space $T_e(D)$. Since both X and D_y are totally geodesic submanifolds of D, the geodesic through e defined by v is contained in both X and D_y, in contradiction with the fact that $|X\cap D_y|=1$. $\square$

3.3. **Orientations.** We fix an orientation on D, as well as an orientation on the vector space $V^- := i\cdot\mathrm{Skew}_{n+1}\subset\mathfrak{s}_{n+1}$. As described in [16, §2], these choices induce an orientation on each submanifold D_y for $y\in\mathfrak{s}_{n+1}$. Note that if V is a quadratic space of signature (p,q) with $p\geq 2$ and $q\geq 1$, then the set of positive length vectors $V_{>0}\subset V$ forms a connected manifold. Thus there are only two conventions for orienting the family $\{D_y\}_{y\in\mathfrak{s}_{n+1},\,y\neq 0}$ such that the orientations vary continuously in y. The definition in [16, §2] can be understood as fixing one of them.

Given an orientation on X, let $[X]$ denote the resulting oriented manifold. The transversality statement in Proposition 3.1 allows to consider the topological intersection numbers $[X]\cdot_{[D]}[D_y]\in\{\pm1\}$ whenever y is regular semi-simple. Recall that we fixed the transfer factor ε.

Proposition 3.2. *There exists an orientation on X such that for every regular semi-simple element $y\in\mathfrak{s}_{n+1}$,*

$$[X]\cdot_{[D]}[D_y] = \begin{cases}\varepsilon(y) & \text{if } y \text{ matches to signature } (n,0)\\ 0 & \text{otherwise.}\end{cases} \tag{3.9}$$

Proof. Let $z\in\mathfrak{s}_{n+1,\mathrm{rs}}$ match to signature $(n,0)$. Then X intersects D_z by Proposition 3.1. Fix the orientation on X such that $[X]\cdot_{[D]}[D_z]=\varepsilon(z)$. We claim that, with this choice, (3.9) holds for all $y\in\mathfrak{s}_{n+1,\mathrm{rs}}$.

By Proposition 3.1, the intersection number $[X]\cdot_{[D_y]}[D]$ is non-zero precisely for $y\in\mathfrak{S}$, the subset of $y\in\mathfrak{s}_{n+1,\mathrm{rs}}$ that match to signature $(n,0)$. This set has two connected components by Lemma 2.3, and these are interchanged by $G\setminus G^0$. By definition of the transfer factor, we have $\varepsilon(g\cdot y)=\eta(g)\varepsilon(y)$ for all $g\in G$. Our task is hence to show that we have the identity

$$[X]\cdot_{[D]}[D_{g\cdot y}] = \eta(g)\cdot[X]\cdot_{[D]}[D_y]$$

for one choice of $y \in \mathfrak{S}$.

The action of G^0 on X and D extends to an action of G by extending the formulas in (3.1) and (3.2):

$$G \times X \longrightarrow X, \quad g \cdot H = {}^t g^{-1} \cdot H g^{-1}$$
$$G \times D \longrightarrow D, \quad g \cdot W = g W g^{-1}.$$

For all $g \in G$, invariance of intersection numbers under isomorphisms implies that

$$[X] \cdot_{[D]} [D_y] = (g \cdot [X]) \cdot_{(g \cdot [D])} (g \cdot [D_y])$$

where the terms $(g \cdot [M])$ denote the image $g(M)$ together with their pushforward orientation.

We consider the specific element $\sigma = \mathrm{diag}(-1, 1, \ldots, 1) \in G$ which stabilizes the base point $e = 1_n \in X$. It is clear from definitions that $\sigma \cdot D_y = D_{\sigma y}$. We need to understand how σ interacts with the orientations on X, D, D_y and $D_{\sigma y}$.

(1) The tangent space $T_e(X)$ is Sym_n and conjugation by σ on Sym_n has determinant $(-1)^{n-1}$. So we obtain

$$\sigma \cdot [X] = (-1)^{n-1} [X]. \tag{3.10}$$

(2) The tangent space $T_e(D)$ is $\mathrm{Hom}(i \cdot \mathrm{Skew}_{n+1}, i \cdot \mathrm{Sym}_{n+1})$. Conjugation by σ acts with determinant

$$((-1)^n)^{\dim(\mathrm{Skew}_{n+1})} \cdot ((-1)^n)^{\dim(\mathrm{Sym}_{n+1})} = (-1)^{n(n+1)(n+1)} = 1.$$

So we find

$$\sigma \cdot [D] = [D]. \tag{3.11}$$

(3) Finally, let $y \in \mathfrak{S}$ be such that $e \in D_y$. The orientation on $T_e(D_y)$ is defined as follows. Recall that we have fixed a reference orientation on $i \cdot \mathrm{Skew}_{n+1}$. Orient the line $\langle y \rangle$ by declaring y to be positive. Consider the induced orientation on $\mathrm{Hom}(i \cdot \mathrm{Skew}_{n+1}, \langle y \rangle)$. Finally, orient the space $T_e(D_y) = \mathrm{Hom}(i \cdot \mathrm{Skew}_{n+1}, \langle y \rangle^{\perp})$ by requiring that the direct sum decomposition

$$\mathrm{Hom}(i \cdot \mathrm{Skew}_{n+1}, i \cdot \mathrm{Sym}_{n+1}) = T_e(D_y) \oplus \mathrm{Hom}(i \cdot \mathrm{Skew}_{n+1}, \langle y \rangle) \tag{3.12}$$

is compatible with orientations. Moreover, σ acts on $i \cdot \mathrm{Skew}_{n+1}$ via a transformation with determinant $(-1)^n$. Thus, by (2) and (3.12), we conclude

$$\sigma \cdot [D_y] = (-1)^n [D_{\sigma y}]. \tag{3.13}$$

Taking (3.10), (3.11) and (3.13) together, we obtain

$$[X] \cdot_{[D]} [D_{\sigma y}] = -[X] \cdot_{[D]} [D_y] \tag{3.14}$$

as we needed to show.

□

4. Schwartz functions

4.1. **Mathai–Quillen formalism.** We begin by recalling a general formalism of Mathai and Quillen [18]. Our presentation here follows [2]. Assume that M is an oriented manifold, and let C^∞_M (resp. $\Omega^\bullet_M$) denote the spaces of smooth functions (resp. differential forms) over M. Tensor products in the following are meant as smooth vector bundles over M, meaning over C^∞_M.

Suppose that $(E, (\ ,\ ))$ is an oriented metrized vector bundle of rank r over M, and that $\nabla \colon E \to \Omega^1_M \otimes E$ is a connection that is compatible with the metric in the sense that

$$d(s_1, s_2) = (\nabla s_1, s_2) + (s_1, \nabla s_2).$$

Let $\kappa\colon E \to \Omega^2_M \otimes E$ denote the curvature, which is a section of $\Omega^2_M \otimes \mathcal{E}nd(E)$. From the compatibility of ∇ with the metric it is immediate that

$$(\kappa(s_1), s_2) + (s_1, \kappa(s_2)) = 0$$

for all sections $s_1, s_2\colon M \to E$. This means that κ defines a section $\kappa : M \to \Omega^2_M \otimes \mathfrak{so}(E)$. We identify $\wedge^2 E \xrightarrow{\sim} \mathfrak{so}(E)$ by

$$s_1 \wedge s_2 \longmapsto [v \mapsto (s_1, v)s_2 - (s_2, v)s_1]$$

and view κ as a section

$$\kappa\colon M \longrightarrow \Omega^2_M \otimes (\wedge^2 E). \tag{4.1}$$

Consider the bigraded (non-commutative) algebra $\mathcal{A} = \Omega^\bullet_M \otimes (\wedge^\bullet E)$ with sign convention

$$(\sigma_1 \otimes \tau_1) \cdot (\sigma_2 \otimes \tau_2) = (-1)^{\deg(\tau_1)\deg(\sigma_2)} (\sigma_1 \wedge \sigma_2) \otimes (\tau_1 \wedge \tau_2).$$

The last piece of notation we need is the *Berezin integral*

$$\{-\}\colon \mathcal{A} \longrightarrow \Omega^\bullet_M.$$

It is given by composing the projection $\mathcal{A} \to \Omega^\bullet_M \otimes (\wedge^{\mathrm{top}} E)$ with the (unique) trivialization $(\wedge^{\mathrm{top}} E) \xrightarrow{\sim} M \times \mathbb{R}$ that preserves orientation and metric.

Now suppose that $s\colon M \to E$ is a section. Consider the three elements

$$\begin{aligned} |s|^2 &:= (s,s) \in C^\infty_M, \\ \nabla(s) &\in \Omega^1_M \otimes E \\ \kappa &\in \Omega^2_M \otimes (\wedge^2 E). \end{aligned} \tag{4.2}$$

all viewed in the algebra $\mathcal{A}$.

Definition 4.1. Let $r = \mathrm{rank}(E)$. The *Mathai–Quillen form* of s defined by the above data is the r-form

$$\psi_s := (-1)^{r(r-1)/2} (2\pi)^{-r/2} \big\{ e^{-2\pi|s|^2 - 2\sqrt{\pi}\nabla(s) - \kappa} \big\} \ \in \Omega^r_M.$$

Here the exponential is defined by the usual power series, with products taking place in the algebra $\mathcal{A}$. We remark that the normalizing factors are motivated by Theorem 4.7 below.

We denote by $Z(s)$ or $Z_s \subseteq M$ the zero locus of s. Assume in addition that s is a *regular* section, meaning that for all points $x \in Z_s$, the induced map $ds\colon T_x(M) \to E_x$ from tangent space to fiber is surjective. Then Z_s is a submanifold of M of codimension r.

Definition 4.2. Let $[Z_s]$ denote the submanifold Z_s equipped with the following orientation: for any point $x \in Z_s$, the derivative ds defines an isomorphism $N_x \simeq E_x$ where N is the normal bundle of Z_s. The fixed orientation on E pulls back to an orientation of N. As we have also fixed an orientation on M, we define an orientation on Z_s via the identification $\wedge^{top} N_x \otimes \wedge^{top} T_x Z_s \xrightarrow{\sim} \wedge^{top} T_x(M)$.

For a compactly supported differential form $\eta \in \Omega^\bullet_{c,M}$, we define the δ-current

$$\delta_{[Z_s]}(\eta) := \int_{[Z_s]} \eta|_{Z_s}.$$

Proposition 4.3. *Let M be an oriented manifold, E an oriented metrized vector bundle of rank r, and ∇ a compatible connection. Suppose that s is a regular section of E with oriented vanishing locus $[Z_s]$.*

Let ψ_s denote the Mathai–Quillen form, and for a compactly supported differential form $\eta \in \Omega^\bullet_{c,M}$ consider the current

$$[\psi_s](\eta) := \int_M \psi_s \wedge \eta.$$

Note that for $t \in \mathbb{R}_{>0}$, the section ts is again regular, and induces the same orientation on $Z_s = Z_{ts}$. We then have

$$\lim_{t\to\infty} [\psi_{ts}] = \delta_{[Z(s)]}.$$

as currents on M.

Proof. This proposition follows from the estimates in [6, Theorem 3.12]; see also [10, Theorem 2.1] for a direct proof in local coordinates. □

We will also have need for the following "transgression formula":

Proposition 4.4. *Let M be an oriented manifold and E an oriented metrized vector bundle, equipped with a compatible connection, as above. For any section s of E, we define the* transgression form

$$\zeta_s := (-1)^{r(r-1)/2}(2\pi)^{-r/2} \left\{ s \wedge e^{-2\pi|s|^2 - 2\sqrt{\pi}\nabla(s) - \kappa} \right\} \in \Omega^{r-1}(M).$$

Then for $t \in \mathbb{R}_{>0}$, we have

$$t\frac{\partial}{\partial t}\psi_{ts} = -2\sqrt{\pi}\, d\zeta_{ts}.$$

Proof. This is proved in [18, §7]; see also [10, Prop. 1.3]. □

Proposition 4.5. *Suppose M is an oriented manifold, E is an oriented vector bundle of rank r, and s is a regular section of E with vanishing locus $[Z_s]$ oriented as per Definition 4.2.*

(1) On $M \setminus Z_s$, the integral

$$g_s := \int_1^\infty \zeta_{ts} \frac{dt}{t} \tag{4.3}$$

defines a smooth form.

(2) The form $g_s \in \Omega^{r-1}(M \setminus Z_s)$ extends to a locally L^1-form on M.

(3) Let $[g_s]$ denote the current given by integration against g_s. Then we have the identity

$$2\sqrt{\pi}\, d[g_s] + \delta_{[Z_s]} = [\psi_s] \tag{4.4}$$

of currents on M.

Proof. These statements follow from analogous estimates to those found in [6, Theorem 3.12], which are presented in greater generality (and under slightly different assumptions) in *loc. cit.* For the convenience of the reader, we give a self-contained argument here.

Let $C = (-1)^{r(r-1)/2}(2\pi)^{-r/2}$. Since $|s|^2$ commutes with both $\nabla(s)$ and κ in the algebra $\mathcal{A}$, we may write

$$\zeta_s = Ce^{-2\pi|s|^2}\{s \wedge e^{-2\sqrt{\pi}\nabla(s) - \kappa}\} \tag{4.5}$$

Note that the expression $\{s \wedge e^{-2\sqrt{\pi}\nabla(s)-\kappa}\}$ is a polynomial expression in $s, \nabla(s)$ and κ. In particular, we may write

$$\zeta_s = \sum_{k=1}^{r} \eta_k(s) e^{-2\pi|s|^2}$$

where $\eta_k(s) \in \Omega^{r-1}(M)$ is homogeneous in s of degree k, i.e. $\eta_k(ts) = t^k \eta_k(s)$ for $t \in \mathbb{R}$. Then, on $M \setminus Z_s$, we have

$$\begin{aligned} g_s &= \sum_{k=1}^{r} \left(\int_1^\infty t^k e^{-2\pi t^2 |s|^2} \frac{dt}{t} \right) \eta_k(s) \\ &= \sum_{k}^{r} |s|^{-k} \left(\int_{|s|}^\infty t^k e^{-2\pi t^2} \frac{dt}{t} \right) \eta_k(s). \end{aligned} \tag{4.6}$$

For each $k > 0$, the integrals appearing above define smooth functions on $M \setminus Z_s$ and the function $|s|^{-k}$ is also smooth on $M \setminus Z_s$. Part (1) follows from these observations.

To prove part (2), it suffices to show that g_s is locally L^1 in a neighbourhood of Z_s. If Z_s is empty, there is nothing to show. Otherwise, fix a point $z \in Z_s$ and a coordinate chart $U \subset M$ around z with coordinates $x_1, \dots x_n$ mapping z to $0 \in \mathbb{R}^n$. We may further assume that there is a local orthonormal frame $e_1, \dots, e_r$ of $E|_U$ such that

$$s = \sum_1^r x_i e_i.$$

In particular, we have $|s|^2 = x_1^2 + \cdots + x_r^2$. Substituting the above expression for s into (4.5), we conclude that upon restriction to U, each $\eta_k(s)$ can be written as a linear combination of the form

$$\eta_k(s)|_U = \sum_{i=1}^r x_i \cdot (\text{smooth form on } U)\,.$$

Moreover, all the integrals appearing in (4.6) are evidently bounded. We are reduced to showing that the functions

$$\frac{x_i}{|s|^k} = \frac{x_i}{(x_1^2 + \cdots + x_r^2)^{k/2}}, \qquad i, k \le r$$

are integrable on a sufficiently small open neighbourhood of $0 \in \mathbb{R}^n$; this latter fact is a straightforward calculus exercise via polar coordinates.

Finally, we prove part (3). Suppose η is a compactly supported form on M. Then

$$\begin{aligned} d[g_s](\eta) = \int_M g_s \wedge d\eta &= \int_M \left(\lim_{t\to\infty} \int_1^t \zeta_{rs} \frac{dr}{r} \right) \wedge d\eta \\ &= \lim_{t\to\infty} \int_M \left(\int_1^t \zeta_{rs} \frac{dr}{r} \right) \wedge d\eta \\ &= \lim_{t\to\infty} \int_M d \left(\int_1^t \zeta_{rs} \frac{dr}{r} \right) \wedge \eta \end{aligned} \tag{4.7}$$

where the interchange of limit and integral in the second equality is justified by the proof of part (2) and the dominated convergence theorem. Applying Proposition 4.4, we have

$$\begin{aligned} d \left(\int_1^t \zeta_{rs} \frac{dr}{r} \right) &= \int_1^t d\zeta_{rs} \frac{dr}{r} \\ &= -\frac{1}{2\sqrt{\pi}} \int_1^t \frac{\partial}{\partial r} \psi_{rs} dr \\ &= -\frac{1}{2\sqrt{\pi}} \left(\psi_{ts} - \psi_s \right). \end{aligned}$$

Substituting this into (4.7) and applying Proposition 4.3 yields the result.

□

4.2. **The Kudla–Millson form and a result of Brancherau.** We begin with a slightly more general situation. Let (V,Q) be a real quadratic space of signature (p,q). Let $V = V^+ \oplus V^-$ be a decomposition into maximal positive and negative, definite subspaces respectively. Let $D(V) = SO(V)^0/SO(V^+) \times SO(V^-)$ be the corresponding symmetric space. As before, there is an identification

$$D(V) \simeq \{z \subset V \mid Q|_z < 0 \text{ and } \dim z = q\}\,.$$

The space $D(V)$ is naturally equipped with a tautological vector bundle $\widetilde{E}$ of rank q; concretely, the fibre $\widetilde{E}_z$ at a point $z \in D$ is simply the space z. We define a metric $(\cdot,\cdot)_{\widetilde{E}}$ on $\widetilde{E}$ by the formula

$$(\widetilde{s},\widetilde{s})_{\widetilde{E}}(z) = -2Q(\widetilde{s}(z)), \qquad \widetilde{s}\colon D(V) \to \widetilde{E}.$$

It is clear that the datum $(\widetilde{E},(\cdot,\cdot)_{\widetilde{E}})$ is naturally $SO(V)^0$-equivariant. Given any $v \in V$, there is a section

$$\widetilde{s}_v\colon D(V) \longrightarrow \widetilde{E}, \qquad \widetilde{s}_v(z) = \mathrm{pr}_z(v)$$

i.e. for $z \in D(V)$, we take the orthogonal projection $\mathrm{pr}_z(v)$ of v onto z. By construction, this section satisfies

$$\gamma^*\widetilde{s}_v = \widetilde{s}_{\gamma^{-1}v} \tag{4.8}$$

for any $\gamma \in SO(V)^0$. Moreover, we have

$$Z(\widetilde{s}_v) = D_v$$

where

$$D_v := \{z \in D(V) \mid z \perp v\}$$

is the Kudla–Millson cycle from Section §3.2.

Furthermore, fix an orientation on $D(V)$ and $\widetilde{E}$. We note that as $D(V)$ is connected, an orientation on $\widetilde{E}$ is determined by the choice of an orientation on any single fibre $\widetilde{E}_z$. Such a choice determines an orientation of D_y, as in [16, §2]. On the other hand, if $Q(v) > 0$, then it is straightforward to check that $\widetilde{s}_v$ is regular, which in turn induces an orientation on $[Z(\widetilde{s}_y)]$ via Definition 4.2. Unwinding the definitions, one can verify that the two constructions coincide, i.e.

$$[Z(\widetilde{s}_y)] = [D_y]. \tag{4.9}$$

Finally, the bundle $\widetilde{E}$ is equipped with a natural connection $\nabla_{\widetilde{E}}$ called the *Maurer-Cartan connection*; see e.g. [7, §3.1] for details. This connection is compatible with the metric on $\widetilde{E}$, and is $SO(V)^0$-equivariant.

Having specified the requisite data, we now have the corresponding Mathai–Quillen form

$$\psi_{\widetilde{s}_v} \in \Omega^q(D(V)) \tag{4.10}$$

as in Definition 4.1.

On the other hand, Kudla and Millson have constructed an explicit differential form, satisfying a natural Thom form property with respect to D_y. We briefly recall the construction, referring to [7, §2] for a more detailed discussion. We begin by noting that there is a canonical identification

$$\mathfrak{P} := T_e(D(V)) \simeq \mathrm{Hom}(V^-, V^+).$$

Note also that the linear action of $O(V)$ on V induces an action on $D(V)$, and the maximal compact subgroup $\widetilde{K} := O(V^+) \times O(V^-)$ acts on $\mathfrak{P}$ by post- and pre-composition in the natural way.

Fix orthonormal bases $x_1, \ldots, x_p$ for V^+ and $x_{p+1}, \ldots, x_{p+q}$ for V^-. These induce a basis $\{X_{ij}\}$ for $\mathfrak{P}$, with $1 \le i \le p$ and $p+1 \le j \le p+q$. Concretely, $X_{ij}(x_k) = \delta_{jk} x_i$. Let ω_{ij} denote the dual basis, and define the Howe operator

$$H \colon \mathcal{S}(V) \otimes_{\mathbb{C}} \wedge^{\bullet}\mathfrak{P}^* \longrightarrow \mathcal{S}(V) \otimes_{\mathbb{C}} \wedge^{\bullet+q}\mathfrak{P}^*$$

by the formula

$$H := 2^{-q} \cdot \prod_{j=p+1}^{p+q} \sum_{i=1}^{p} \left[\left((x_i - \frac{1}{2\pi} \frac{\partial}{\partial x_i} \right) \otimes A_{ij} \right]$$

where $\mathcal{S}(V)$ is the space of Schwartz functions on V, and A_{ij} is left multiplication by ω_{ij} .

Definition 4.6 ([16, §5])**.** The Kudla–Millson form φ_{KM} is the q-form obtained by applying the Howe operator to the Gaussian:

$$\varphi_{\mathrm{KM}} := H \cdot e^{-\pi(\sum_{i=1}^{p+q} x_i^2)} \in \mathcal{S}(V) \otimes (\wedge^q \mathfrak{P}^*).$$

This form satisfies the following equivariance property. Let $\nu \colon O(V) \to \{\pm 1\}$ denote the spinor norm. Recall that this is the unique character whose restriction to $\widetilde{K} = O(V^+) \times O(V^-)$ is given by $\nu(k^+, k^-) = \det(k^-)$. We then have

$$\varphi_{\mathrm{KM}}(k^{-1}v) = \nu(k) \cdot k^*(\varphi_{\mathrm{KM}}(v)) \in \wedge^q \mathfrak{P}^* \tag{4.11}$$

for all $v \in V$ and $k \in \widetilde{K}$, see [13, Theorem 3.1].

In particular, φ_{KM} lies in the space of invariants $(\mathcal{S}(V) \otimes \wedge^q \mathfrak{P}^*)^{SO(V^+) \times SO(V^-)}$. There is a natural isomorphism

$$[\mathcal{S}(V) \otimes \wedge^q \mathfrak{P}^*]^{SO(V^+) \times SO(V^-)} \xrightarrow{\sim} [\mathcal{S}(V) \otimes \Omega^q(D(V))]^{SO(V)^0}, \qquad \eta \longmapsto \widetilde{\eta}$$

determined by the relation $\widetilde{\eta}|_e = \eta$. We define

$$\widetilde{\varphi}_{\mathrm{KM}} \in [\mathcal{S}(V) \otimes \Omega^q(D(V))]^{SO(V)^0}$$

to be the image of φ_{KM} under this isomorphism.

We now have two constructions of differential forms associated to a vector $v \in V$. The following theorem of Brancherau asserts that the two constructions essentially coincide:

Theorem 4.7 ([7])**.** *For any $v \in V$, we have*

$$\widetilde{\varphi}_{\mathrm{KM}}(v) = 2^{-q/2} e^{-2\pi Q(v)} \psi_{\widetilde{s}_v}.$$

□

Now we specialize the notation to our case of interest. Take $V = \mathfrak{s}_{n+1}$ and $D = D(\mathfrak{s}_{n+1})$ as before. Let $X = \mathrm{Sym}_n(\mathbb{R})_{>0}$ and consider the map $\alpha \colon X \to D$ as in (3.4). In fact, it will be more convenient to work on the group $B \subset G$ of upper triangular matrices. Consider the surjective map

$$B \longrightarrow X = \mathrm{Sym}_n^{>0}, \qquad b \longmapsto {}^t b^{-1} b^{-1} \tag{4.12}$$

which is a finite covering map of degree 2^n, and let β be the composition

$$\beta \colon B \longrightarrow X \xrightarrow{\alpha} D.$$

Pulling back via β, we obtain the bundle $E := \beta^*\widetilde{E}$, equipped with the pullback metric $(\cdot,\cdot)_E$ and connection ∇_E, as well as a pullback section $s_y = \beta^*\widetilde{s}_y$ for any $y \in \mathfrak{s}_{n+1}$. We let

$$\psi_y := \psi_{s_y} \in \Omega^{n(n+1)/2}(B). \tag{4.13}$$

and

$$\zeta_y := \zeta_{s_y} \in \Omega^{n(n+1)/2-1}(B) \tag{4.14}$$

denote the Mathai–Quillen and transgression forms attached to this data. Note that $\dim(B) = n(n+1)/2$, i.e. ψ_y is a form of top degree on B. Moreover, it is clear from the construction that these forms are functorial in the data defining them, i.e. we have

$$\psi_y = \beta^*\psi_{\widetilde{s}_y}, \qquad \zeta_y = \beta^*\zeta_{\widetilde{s}_y}.$$

Remark 4.8. The role that Theorem 4.7 plays in our present work is as follows. On the one hand, as evident in Definition 4.6, it is straightforward to write down explicit formulas for the Kudla–Millson form. On the other hand, we may apply the general machinery of the Mathai–Quillen formalism, in particular the current equation (4.4), to deduce geometric properties that are not immediately evident from the construction.

4.3. **Schwartz forms.** We will employ the terminology of Schwartz forms on real affine algebraic varieties, which is a special case of that of Nash manifolds discussed in [1].

Definition 4.9. Let M be a real affine algebraic smooth variety, and let $C^\infty(M)$ and $\Omega^\bullet(M)$ denote the space of smooth functions, and smooth differential forms, respectively. A function $f \in C^\infty(M)$ is called a *Schwartz function* if for every algebraic differential operator D, the function Df is bounded on M (cf. [1, Corollary 4.1.3]). The space of *Schwartz (differential) forms* is the subspace of $\Omega^\bullet(M)$ spanned by elements of the form $f\omega$ where f is a Schwartz function and ω is an algebraic differential form.

Note that if $M = \mathbb{R}^m$, we recover the usual notion of Schwartz functions, i.e. rapidly decaying functions on $\mathbb{R}^m$ such that all partial derivatives of all orders are also rapidly decaying.

Lemma 4.10. *Let M be an oriented real affine algebraic variety, and let $m = \dim(M)$.*

(1) If $\Phi \in \Omega^m(M)$ is a Schwartz form of top degree, then the integral

$$\int_M \Phi$$

exists (i.e. is finite).

(2) If $\Phi \in \Omega^{m-1}(M)$ is a Schwartz form, then

$$\int_M d\Phi = 0.$$

The same conclusions hold if M is replaced by any connected component M' of M.

Proof. Both claims follow from straightforward calculus arguments when $M = \mathbb{R}^m$.

In general, suppose M' is a connected component of M, and $\Phi \in \Omega^\bullet(M')$. There exists a finite open cover $M' = U_1 \cup \cdots \cup U_j$ such that each U_i is isomorphic to $\mathbb{R}^m$ as a Nash manifold. Applying a partition of unity, as in [1, Theorem 4.4.1], there are Schwartz forms $\Phi_i \in \Omega^\bullet(U_i)$ such that

$$\Phi = \sum \Phi_i.$$

In this way, we reduce both claims to the case $M = \mathbb{R}^m$. ∎

We now apply this discussion to the case $M = B$, the group of upper triangular matrices, and the metrized bundle E, equipped with its connection ∇_E. To this end, we first endow E with an algebraic structure.

Recall that $\rho\colon \mathrm{GL}_n(\mathbb{R}) \to O(\mathfrak{s}_{n+1})$ denotes the representation (3.3), and recall that we had fixed a decomposition $\mathfrak{s}_{n+1} = V^+ \oplus V^-$ with $V^- := i \cdot \mathrm{Skew}_{n+1}$. By definition, we have $\beta(b) = \rho(b)(V^-) \in D$ for any $b \in B$. By equivariance, E is a trivial vector bundle with trivialization

$$\mathrm{triv}\colon B \times V^- \xrightarrow{\sim} E, \qquad (b, v^-) \longmapsto (b, \rho(b)v^-). \tag{4.15}$$

Using this isomorphism, we identify E with the $\mathbb{R}$-points of the trivial algebraic vector bundle $B \times V^-$. That is, a section $s : B \to E$ is algebraic if and only if the function $f : B \to V^-$ defining $\mathrm{triv}^{-1} \circ s$ is algebraic. An equivalent way to phrase this definition is declaring B-invariant sections of E algebraic.

Lemma 4.11. *(1) For any $y \in \mathfrak{s}_{n+1}$, the sections s_y and $\nabla_E(s_y)$ are algebraic.*

(2) If y is regular semi-simple, then s_y is regular in the sense of Section 4.1.

Proof. It is a direct consequence of definitions that, in terms of (4.15), $s_y = \beta^*\widetilde{s}_y$ corresponds to the function

$$f_y\colon B \longrightarrow V^-, \qquad f_y(b) = \mathrm{pr}_{V^-}\left(\rho(b)^{-1}(y)\right)$$

and that $\nabla_E(s_y)$ corresponds to $d_B f_y$, [I don't understand the statement about $\nabla_E(s_y)$.] both of which are evidently algebraic.

For a regular semisimple element y, the regularity of s_y was already shown in Proposition 3.1 . ☐

Proposition 4.12. *(1) Suppose that $y \in \mathfrak{s}_{n+1}$ is regular semi-simple. Then ψ_y and ζ_y are Schwartz forms on B.*

(2) Let $\rho \in C_c^\infty(B)$ be a function such that $\rho \equiv 1$ in a neighbourhood of Z_s, and let $f = 1 - \rho$. Then fg_s is a Schwartz form.

Proof. In the sequel, use the abbreviations

$$s = s_y, \qquad q = n(n+1)/2, \quad \text{and} \quad C = (-1)^{q(q-1)/2}(2\pi)^{-q/2}.$$

(1) By definition, we have

$$\psi_y = C\{e^{-2\pi|s|^2 - 2\sqrt{\pi}\nabla(s) - \kappa}\} \qquad \text{and} \qquad \zeta_y = C\{s \wedge e^{-2\pi|s|^2 - 2\sqrt{\pi}\nabla(s) - \kappa}\}.$$

Since $|s|^2$ commutes with both $\nabla(s)$ and κ in the algebra $\mathcal{A}$, we may rewrite these forms as

$$\psi_y = C\, e^{-2\pi|s|^2}\{e^{-2\sqrt{\pi}\nabla(s) - \kappa}\}, \qquad \zeta_y = Ce^{-2\pi|s|^2}\{s \wedge e^{-2\sqrt{\pi}\nabla(s) - \kappa}\}$$

Note that the differential forms $\{e^{-2\sqrt{\pi}\nabla(s) - \kappa}\}$ and $\{s \wedge e^{-2\sqrt{\pi}\nabla(s) - \kappa}\}$ appearing above can be expressed as polynomial expressions in $s, \nabla(s)$ and κ, and hence are algebraic, by Lemma 4.11. It therefore will suffice to show that the function

$$\Phi(b) := e^{-2\pi|s(b)|^2}$$

is a Schwartz function on B. By definition of s and the metric on E, this function is given by

$$\Phi(b) = e^{2\pi Q(\mathrm{pr}_{V^-}(\rho(b)^{-1}(y))}.$$

We multiply it with the constant $e^{-\pi Q(y)}$. Recall that Q is invariant along the G-orbits of $\mathfrak{s}_{n+1}$, implying $Q(y) = Q(\rho(b)^{-1}(y))$ for all $b \in B$. Moreover, from by the orthogonal decomposition $V = V^+ \oplus V^-$, we have

$$Q(\rho(b)^{-1}(y)) = Q\big(\mathrm{pr}_{V^+}(\rho(b)^{-1}(y))\big) + Q\big(\mathrm{pr}_{V^-}(\rho(b)^{-1}(y))\big).$$

Hence we see

$$\begin{aligned} e^{-\pi Q(y)}\Phi(b) &= e^{-\pi[Q(\mathrm{pr}_{V^+}(\rho(b)^{-1}(y)))-Q(\mathrm{pr}_{V^-}(\rho(b)^{-1}(y)))]} \\ &= e^{-\pi Q^*(\rho(b)^{-1}(y))} \end{aligned}$$

for the positive definite quadratic form $Q^* = Q|_{V^+} - Q|_{V^-}$. (In fact, this is the Siegel majorant.) In other words, we consider the Schwartz function $\Phi'(v) = e^{-\pi Q^*(v)}$ on $\mathfrak{s}_{n+1}$, and pull it back to B under the orbit map

$$B \longrightarrow \mathfrak{s}_{n+1}, \qquad b \longmapsto b \cdot y.$$

Since y is regular semi-simple, the stabilizer of y under the action of B is trivial, and the orbit $B \cdot y$ is Zariski closed in $\mathfrak{s}_{n+1}$. Moreover, the action $B \times \mathfrak{s}_{n+1} \to \mathfrak{s}_{n+1}$ is an algebraic map, and in particular, a closed immersion. Thus we obtain an identification $B \simeq B \cdot y$ as a closed Nash submanifold of $\mathfrak{s}_{n+1}$. As the restriction of a Schwartz function on a Nash manifold to a closed Nash submanifold is again a Schwartz function, we see that $\Phi = \Phi'|_{B\cdot y}$ is Schwartz, concluding the proof.

(2) Suppose $f \in C^\infty(M)$ satisfies $|f| < C$ for some constant C and $f \equiv 0$ on an open neighbourhood U of Z_s. Using (4.6), we may write

$$fg_s = \sum_{k=0}^{r-1} f\left(\int_1^\infty t^k e^{-2\pi t^2|s|^2}\frac{dt}{t}\right)\eta_k(s)$$

where each $\eta_k(s)$ is a polynomial expression in s, $\nabla(s)$ and κ. In particular, each $\eta_k(s)$ is algebraic.

Moreover, there exists a constant $c > 0$ such that

$$|s(b)|^2 > c \qquad \text{for all } b \in B \setminus U.$$

We then have that for any $k \geq 0$,

$$\begin{aligned} \left| f\int_1^\infty t^k e^{-2\pi t^2|s|^2}\frac{dt}{t}\right| &< C\int_1^\infty t^k e^{-2\pi t^2|s|^2}\frac{dt}{t} \\ &< C\int_1^\infty t^k e^{-2\pi t^2|(\frac{1}{2}|s|^2+\frac{c}{2})}\frac{dt}{t} \\ &< Ce^{-\pi|s|^2}\int_1^\infty t^k e^{-\pi c t^2}\frac{dt}{t} \end{aligned}$$

The integral is finite, and the function $e^{-\pi|s|^2}$ is Schwartz, as in part (1). This concludes the proof. □

Let $B^0 \subset B$ denote the connected component of the identity, so that the map $B \to X$ restricts to an isomorphism

$$B^0 \xrightarrow{\sim} X. \tag{4.16}$$

This isomorphism also identifies

$$B^0 \cap Z_{s_y} \xrightarrow{\sim} X \cap D_y = \begin{cases} pt & \text{if } y \text{ matches to signature } (n,0) \\ \emptyset & \text{otherwise} \end{cases}$$

cf. Proposition 3.1. Furthermore, in Section 3.3, we had fixed orientations on D and on $V^- = i \cdot \mathrm{Skew}_{n+1}$, which in turn determined an orientation on D_y, as well as on X via Proposition 3.2. The orientation on V^- determines an orientation on the bundle $\widetilde{E}$, and we have

$$[Z_{s_y}] = [D_y],$$

where $[Z_{s_y}]$ is oriented according to Definition 4.2. On the other hand, using (4.16) to transfer the orientation on X to B^0, we have

$$\deg\left([Z_{s_y}]\right) = [X] \cdot_{[D]} [D_y] = \begin{cases} \varepsilon(y) & \text{if } y \text{ matches to signature } (n,0) \\ 0 & \text{otherwise.} \end{cases} \tag{4.17}$$

Theorem 4.13. *Let $y \in \mathfrak{s}_{n+1}$ be a regular semi-simple element. Then*

$$\int_{B^0} \psi_y = \begin{cases} \varepsilon(y) & \textit{if } y \textit{ matches to signature } (n,0) \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If B^0 were compact, this identity would follow immediately from Proposition 4.5 by evaluating (4.4) at the constant function 1. In our case however, we will need to be a bit more indirect.

Let us again abbreviate $s = s_y$. We begin by fixing a sequence of successively relatively compact open neighbourhoods

$$U_1 \subset U_2 \subset \cdots$$

of Z_s such that $\cup U_k = B^0$. (If Z_s is empty, we fix an arbitrary family of nested relatively compact open sets exhausting B^0.) We choose a family of compactly supported functions $\rho_k \in C_c^\infty(B_0)$ such that

$$|\rho_k(b)| \leq 1 \qquad \text{for all } b \in B^0$$

and

$$\rho_k \equiv 1 \qquad \text{on } U_k.$$

Then, by Proposition 4.5, we have

$$\begin{aligned} \int_{B^0} \psi_s &= \lim_{k\to\infty} \int_{B^0} \psi_s \rho_k \\ &= \lim_{k\to\infty} \delta_{[Z(s)]}(\rho_k) + 2\sqrt{\pi} \int_{B^0} g_s \wedge d\rho_k. \end{aligned}$$

By construction, we also have for all k that

$$\delta_{[Z(s)]}(\rho_k) = \deg([Z_s]) = \begin{cases} \varepsilon(y) & \text{if } y \text{ matches to signature } (n,0) \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, let $f \in C^\infty(B^0)$ be such that $f \equiv 0$ on U_1 and $f \equiv 1$ on $B^0 \setminus U_2$. If $k \geq 3$, then $d\rho_k$ is supported on $B^0 \setminus U_2$, so

$$\int_{B^0} g_s \wedge d\rho_k = \int_{B^0} (f g_s) \wedge d\rho_k = \int_{B^0} d(f g_s) \wedge \rho_k.$$

By Proposition 4.12 and Lemma 4.10, we have

$$\lim_{k\to\infty} \int_{B^0} d(f g_s) \wedge \rho_k = \int_{B^0} d(f g_s) = 0,$$

concluding the proof of the theorem. $\square$

4.4. **Differential forms and orbital integrals.** As a final step, we interpret Theorem 4.13 in terms of orbital integrals. Consider the Iwasawa decomposition $G = B^0 \cdot O(n)$. We obtain a decomposition $dg = db\,dk$ of measures, where dg is our fixed Haar measure on G, where dk is the Haar measure on $O(n)$ normalized to have total volume 1, and where db is the left Haar measure determined by the integral formula

$$\int_G f(g)dg \;=\; \int_{B^0}\int_{O(n)} f(bk)\,dk\,db$$

for any integrable function f.

Next, recall that we have fixed an orientation on $B^0 \simeq X$ as in Section 3.3. Let $\widetilde{\omega} \in \Omega^{top}(B)$ be a left-invariant (algebraic, in particular) differential form of top degree whose restriction to B^0 is positive, and induces the measure db; i.e. for any integrable function f on B, we have

$$\int_{B^0} f\widetilde{\omega} = \int_{B^0} f(b)db.$$

Let $\mathfrak{b} = \mathrm{Lie}(B)$, and let $\omega \in \det(\mathfrak{b})$ be the value $\omega = \widetilde{\omega}_e$ of $\widetilde{\omega}$ at the identity element. Consider the pullback map induced by $\beta : B \to SO(\mathfrak{s}_{n+1})$,

$$\beta^* : \mathcal{S}(\mathfrak{s}_{n+1}) \otimes \wedge^{n(n+1)/2}(\mathfrak{P}^*) \longrightarrow \mathcal{S}(\mathfrak{s}_{n+1}) \otimes \det(\mathfrak{b}),$$

and define $\Phi \in \mathcal{S}(\mathfrak{s}_{n+1})$ by the identity

$$\beta^*(\varphi_{\mathrm{KM}}) = \Phi \otimes \omega.$$

Lemma 4.14. *For $k \in O(n)$ and $y \in \mathfrak{s}_{n+1}$, we have $\Phi(k^{-1} \cdot y) = \eta(k)\Phi(y)$.*

Proof. Let $V^+ = i \cdot \mathrm{Sym}_{n+1}$ and $V^- = i \cdot \mathrm{Sym}_{n+1}$ again denote our usual choice of maximal definite subspaces of $\mathfrak{s}_{n+1}$. The composition

$$O(n) \xrightarrow{\alpha} O(V^+) \times O(V^-) \xrightarrow{\nu} \{\pm 1\}$$

equals η^n, because the element $\sigma = \mathrm{diag}(-1, 1, \ldots, 1)$ acts with determinant $(-1)^n$ on V^-. Moreover, the identification $B_0 \xrightarrow{\sim} X$ induces an isomorphism $\mathfrak{b} \xrightarrow{\sim} T_e X = \mathrm{Sym}_n$, and hence via composition an action of $O(n)$ on $\mathfrak{b}$. Its determinant, which is simply the determinant of $O(n)$ acting on Sym_n, is η^{n-1}.

From the invariance property in (4.11), we obtain for general $k \in O(n)$ that

$$\begin{aligned}\Phi(k^{-1}v) \otimes \omega &= \eta(k)^n \cdot \Phi(v) \otimes (k^*\omega)\\ &= \eta(k)\Phi(v).\end{aligned}$$

□

We now state our main theorem:

Theorem 4.15. *Suppose $y \in \mathfrak{s}_{n+1}$ is regular semi-simple. Then*

$$2^{n(n+1)/4}\,\mathrm{Orb}(y, \Phi) = \begin{cases} e^{-2\pi Q(y)} & \text{if } y \text{ matches to signature } (n, 0) \\ 0 & \text{otherwise.}\end{cases}$$

In other words, $2^{-n(n+1)/4}\Phi$ is a Gaussian test function.

Proof. By definition, we have

$$\begin{aligned}\mathrm{Orb}(y,\Phi) &= \varepsilon(y)\int_G \Phi(g^{-1}\cdot y)\,\eta(g)\,dg\\ &= \varepsilon(y)\int_{B^0}\int_{O(n)} \Phi(k^{-1}b^{-1}\cdot y)\,\eta(k)\,dkdb\\ &= \varepsilon(y)\int_{B^0}\Phi(b^{-1}\cdot y)db\end{aligned}$$

where in the last line, we use the fact that $\Phi(k\cdot y)\eta(k)=\Phi(y)$, as in Lemma 4.14.

Applying our conventions on measures, as well as Theorem 4.7, we have

$$\begin{aligned}\int_{B^0}\Phi(b^{-1}\cdot y)db &= \int_{B^0}\Phi(b^{-1}\cdot y)\,\widetilde{\omega}\\ &= \int_{B^0}\beta^*(\widetilde{\varphi}_{\mathrm{KM}}(y))\\ &= 2^{-n(n+1)/4}e^{-2\pi Q(y)}\int_{B^0}\psi_y.\end{aligned}$$

The result now follows from Theorem 4.13. □

Example 4.16 (Case $n=1$)**.** In this situation our construction recovers the test function used in [26, §12]. Consider $G=\mathbb{R}^\times$ with Haar measure dt/t. We denote the coordinates on $\mathfrak{s}_2$ by

$$\mathfrak{s}_2 = i\cdot\left\{\begin{pmatrix} a & y_1\\ y_2 & d\end{pmatrix}\middle|\, a,d,y_1,y_2\in\mathbb{R}\right\}.$$

Then a and d are G-invariant. We pick a transfer factor ε that is positive whenever y_1 and y_2 are positive. The quadratic form $Q(y)=-\mathrm{tr}(y^2)$ is give by

$$Q(y)=a^2+d^2+2y_1y_2.$$

Its definite components are

$$Q_+(y)=a^2+d^2+\frac{(y_1+y_2)^2}{2},\quad Q_-=\frac{(y_1-y_2)^2}{2}.$$

The Siegel–Gaussian is hence

$$e^{-2\pi(Q_++Q_-)}=e^{-2\pi(a^2+d^2+y_1^2+y_2^2)}.$$

The Schwartz function Φ from Theorem 4.15 is

$$\Phi(y)=2^{-1/2}\cdot(y_1+y_2)e^{-2\pi(a^2+d^2+y_1^2+y_2^2)}.$$

The theorem now states that

$$\mathrm{Orb}\left(i\cdot\begin{pmatrix} a & y_1\\ y_2 & d\end{pmatrix},\Phi\right)=e^{-2\pi(a^2+d^2)}\cdot\begin{cases} e^{-4\pi y_1y_2} & \text{if } y_1y_2>0\\ 0 & \text{otherwise.}\end{cases}$$

Example 4.17 (Case $n=2$)**.** We first normalize the Haar measure on $G=\mathrm{GL}_2(\mathbb{R})$. Use the following notation for the coordinates of the Iwasawa decomposition of an element $g\in G$:

$$g=\begin{pmatrix} a_1 & \\ & a_2\end{pmatrix}\cdot\begin{pmatrix} 1 & b\\ & 1\end{pmatrix}\cdot\theta,\quad a_1,a_2\in\mathbb{R}_{>0},\ b\in\mathbb{R},\ \theta\in O(2).$$

Then fix the measure as

$$dg=\frac{da_1\,da_2\,db\,d\theta}{a_1a_2},\qquad \int_{O(2)}d\theta=1.$$

We denote the coordinates on $\mathfrak{s}_3$ by

$$\mathfrak{s}_3 = i \cdot \left\{ \begin{pmatrix} y_{11} & y_{12} & v_1 \\ y_{21} & y_{22} & v_2 \\ w_1 & w_2 & d \end{pmatrix} \middle| \text{all entries in } \mathbb{R} \right\}.$$

The entry d is G-invariant. We pick a transfer factor ε that is positive on elements of the form (2.7) with $\lambda_1 > \lambda_2$, $\mu_1 > 0$, and $\mu_2 > 0$. The quadratic form $Q(y) = -\mathrm{tr}(y^2)$ is given by

$$Q(y) = y_{11}^2 + y_{22}^2 + d^2 + 2(y_{12}y_{21} + v_1w_1 + v_2w_2).$$

Its definite components are

$$\begin{aligned} Q_+(y) &= y_{11}^2 + y_{22}^2 + d^2 + \frac{(y_{12}+y_{21})^2 + (v_1+w_1)^2 + (v_2+w_2)^2}{2} \\ Q_-(y) &= \frac{(y_{12}-y_{21})^2 + (v_1-w_1)^2 + (v_2-w_2)^2}{2}. \end{aligned}$$

The Siegel–Gaussian is hence

$$e^{-2\pi(Q_+ + Q_-)} = e^{-2\pi(y_{11}^2 + \ \dots \ + d^2)}$$

where the exponent involves the sum of the squares of all 9 entries of y. The Schwartz function Φ from Theorem 4.15 is

$$\begin{aligned} \Phi(y) = \Bigg[&\sqrt{2} \cdot (y_{11} - y_{22})(v_1 + w_1)(v_2 + w_2) \\ &- \frac{1}{\sqrt{2}}(y_{12} + y_{21}) \left((v_1 + w_1)^2 - (v_2 + w_2)^2 \right) \Bigg] \cdot e^{-2\pi(Q_+ + Q_-)}. \end{aligned}$$

5. From Lie algebra to group

Recall the definition of S_{n+1} from (1.1). Our aim is to prove the following theorem.

Theorem 5.1. *There exists a Gaussian Schwartz function on S_{n+1}. That is, there exists a Schwartz function $\Psi \in \mathcal{S}(S_{n+1})$ such that for all regular semi-simple $\gamma \in S_{n+1}$*

$$\mathrm{Orb}(\gamma, \Psi) = \begin{cases} 1 & \text{if } \gamma \text{ matches to signature } (n, 0) \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Recall that the occurring orbital integral was defined in (1.3), but that we still need to provide the definition of the transfer factor $\epsilon : S_{n+1,\mathrm{rs}} \to \mathbb{C}^\times$. This will be done in the next subsection. The proof of Theorem 5.1 will be given after that (see §5.2).

5.1. **Transfer factors.** We completely follow the conventions in the literature, see [25] and [22]. Fix a character $\eta' : \mathbb{C}^\times \to \mathbb{C}^\times$ (necessarily not quadratic) that extends $\eta : \mathbb{R}^\times \to \{\pm 1\}$. Note that all such characters are of the form $z \mapsto (z/|z|)^m$ with $m \in 1 + 2\mathbb{Z}$. Let us write $S_{n+1,\mathrm{rs}} \subseteq S_{n+1}$ for the subset of regular semi-simple elements. Let e be be the row vector $(0, \dots, 0, 1) \in \mathbb{C}_{n+1}$. Define the transfer factor on S_{n+1} by

$$\begin{aligned} \epsilon : S_{n+1,\mathrm{rs}} &\longrightarrow \mathbb{C}^\times \\ \gamma &\longmapsto \eta'\big(\det(\gamma)^{-\lfloor (n+1)/2 \rfloor} \det\left((e, e\gamma, \dots, e\gamma^n) \right) \big). \end{aligned} \tag{5.2}$$

Here, $(e, e\gamma, \dots, e\gamma^n)$ denotes the matrix with $e\gamma^i$ in the $(i+1)$-th row. Note that unlike in the Lie algebra setting, ϵ is not locally constant. It still satisfies $\epsilon(h\gamma h^{-1}) = \eta(h)\epsilon(\gamma)$ for all $h \in \mathrm{GL}_n(\mathbb{R})$ and $\gamma \in S_{n+1,\mathrm{rs}}$. The definition of (1.3) is now complete.

We would like to compare ϵ with a transfer factor on the Lie algebra. So, again following [25] and [22], we make the following explicit choice:

$$\begin{aligned} \varepsilon : \mathfrak{s}_{n+1,\mathrm{rs}} &\longrightarrow \{\pm 1\} \\ y &\longmapsto \eta\big((-i)^{n(n+1)/2} \cdot \det\big((e, ey, \dots, ey^n)\big)\big). \end{aligned} \tag{5.3}$$

Definition 5.2. Let $\xi \in \mathbb{C}^1$ be an element of norm 1. Define open subsets $S_{n+1,\xi}$ and $\mathfrak{s}_{n+1,\xi}$ by the condition

$$\det(\gamma - \xi) \neq 0 \quad \text{resp.} \quad \det(y - \xi) \neq 0. \tag{5.4}$$

We write $S_{n+1,\mathrm{rs},\xi}$ and $\mathfrak{s}_{n+1,\mathrm{rs},\xi}$ for their subsets of regular semi-simple elements. The Cayley transform (with parameter ξ) is the isomorphism

$$\begin{aligned} c_\xi \colon \mathfrak{s}_{n+1,1} &\xrightarrow{\sim} S_{n+1,-\xi} \\ y &\longmapsto \xi\frac{1+y}{1-y}. \end{aligned}$$

Its inverse is given by $\gamma \mapsto (\gamma - \xi)/(\gamma + \xi)$. Note that c_ξ is equivariant with respect to $\mathrm{GL}_n(\mathbb{R})$-conjugation. In particular, it preserves the property of being regular semi-simple, inducing an isomorphism

$$c_\xi : \mathfrak{s}_{n+1,\mathrm{rs},1} \xrightarrow{\sim} S_{n+1,\mathrm{rs},-\xi}.$$

It was shown in [25, Lemma 3.5] that ε and ϵ are compatible under the Cayley transform. There seems to be a typo in the argument, though, so we give a complete proof.

Lemma 5.3 ([25, Lemma 3.5])**.** *Let $\xi \in \mathbb{C}^1$ be an element of norm 1. Then ε on $\mathfrak{s}_{n+1,\mathrm{rs},1}$ and ϵ on $S_{n+1,\mathrm{rs},-\xi}$ are compatible in the sense that there exists a smooth, algebraic, nowhere vanishing function ρ_ξ on $\mathfrak{s}_{n+1,1}$ such that for all regular semi-simple $y \in \mathfrak{s}_{n+1,\mathrm{rs},1}$,*

$$\epsilon(c_\xi(y)) = \rho_\varepsilon(y) \cdot \varepsilon(y).$$

Proof. Define $\gamma = \xi^{-1}c_\xi(y)$; concretely, $\gamma = (1+y)/(1-y)$. Pulling out factors of ξ row by row in the definition of ϵ, we have

$$\epsilon(c_\xi(y)) = \rho_1(\xi)\,\epsilon(\gamma), \quad \rho_1(\xi) = \eta'(\xi)^{n(n+1)/2-(n+1)\cdot\lfloor (n+1)/2\rfloor}.$$

Next, we have

$$\epsilon(\gamma) = \rho_2(y)\,\eta'\big(\det((e, e\gamma, \dots, e\gamma^n)\big), \quad \rho_2(y) = \eta'(\det(1+y)/\det(1-y))^{-\lfloor (n+1)/2\rfloor}.$$

Note that ρ_2 is a smooth function nowhere vanishing function in $y \in \mathfrak{s}_{n+1,1}$. Set $T = 2y/(1-y)$ and observe that $\gamma = 1 + T$. By elementary row operations, we find

$$\det\big((e, e(1+T), \dots, e(1+T)^n)\big) = \det\big((e, eT, \dots, eT^n)\big).$$

Multiplying with $(1-y)^n$ from the right, and writing $r_3(y) = 2^{n(n+1)/2}\det(1-y)^{-n}$, we have

$$\det\big(e, eT, \dots, eT^n\big) = r_3(y)\,\det\big((e(1-y)^n, ey(1-y)^{n-1}, \dots, ey^{n-1}(1-y), ey^n)\big).$$

By elementary row operations, the last determinant equals

$$\det\big((e, ey, \dots, ey^n)\big),$$

which is also the determinant occurring in (5.3). In summary, we obtain that

$$\epsilon(c_\xi(y)) = \rho_1(\xi) \cdot \rho_2(y) \cdot \eta'(r_3(y)) \cdot \eta'(i)^{n(n+1)/2} \cdot \varepsilon(y) \tag{5.5}$$

which completes the proof. ∎

5.2. **Proof of Theorem 5.1.** We now prove Theorem 5.1. Let Q_{n+1} and $\mathfrak{q}_{n+1}$ denote the $\mathbb{R}$-points of the GIT quotients by GL_n of (the algebraic varieties underlying) S_{n+1} and $\mathfrak{s}_{n+1}$. For every n-dimensional hermitian $\mathbb{C}$-vector space V, these are also the GIT quotients by $U(V)$ of $U(V \oplus \mathbb{C})$ and $\mathfrak{u}(V \oplus \mathbb{C})$. The quotient maps

$$\begin{aligned} \mathrm{inv}: \quad & S_{n+1},\, U(V \oplus \mathbb{C}) && \longrightarrow Q_{n+1}, \\ \mathrm{inv}: \quad & \mathfrak{s}_{n+1},\, \mathfrak{u}(V \oplus \mathbb{C}) && \longrightarrow \mathfrak{q}_{n+1} \end{aligned}$$

realize the matching bijections in (1.2) and (1.5).

Recall that our aim is to construct a Schwartz function $\Psi \in \mathcal{S}(S_{n+1})$ such that (5.1) holds. This construction can be performed locally in the following sense: Let $T \subseteq Q_{n+1}$ be the image $\mathrm{inv}(U(n+1))$, which is compact. Assume that $T \subseteq \cup_{i=1}^r U_i$ is an open covering of T in Q_{n+1} and that $\Psi_i \in \mathcal{S}(S_{n+1})$ are such that for all $\gamma \in \mathrm{inv}^{-1}(U_i)$,

$$\mathrm{Orb}(\gamma, \Psi_i) = \begin{cases} 1 & \text{if } \gamma \text{ matches to signature } (n,0) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\rho_i \subseteq C_c^\infty(U_i)$ be such that $(\sum_{i=1}^r \rho_i)|_T \equiv 1$. Then $\sum_{i=1}^r \rho_i \Psi_i$ satisfies (5.1) and the proof of Theorem 5.1 is complete. Our task is hence to construct the datum $(U_i, \Psi_i)_{i=1}^r$. Since T is compact, it suffices for each $t \in T$ to construct an open neighborhood $U \subseteq Q_{n+1}$ and a Schwartz function Ψ_U satisfying (5.1) for $\gamma \in \mathrm{inv}^{-1}(U)$.

The definition of $S_{n+1,\xi}$ and $\mathfrak{s}_{n+1,\xi}$ by (5.4) in Definition 5.2 is in terms of the G-invariant polynomials $\det(\gamma - \xi)$ and $\det(y - \xi)$. There are hence Zariski open subsets $Q_{n+1,\xi}$ and $\mathfrak{q}_{n+1,\xi}$ of Q_{n+1} (resp. $\mathfrak{q}_{n+1}$) such that

$$S_{n+1,\xi} = \mathrm{inv}^{-1}(Q_{n+1,\xi}), \quad \mathfrak{s}_{n+1,\xi} = \mathrm{inv}^{-1}(\mathfrak{q}_{n+1,\xi}).$$

Choose $\xi \in \mathbb{C}^1$ with $t \in Q_{n+1,\xi}$ and consider the Cayley transform

$$c_\xi : \mathfrak{s}_{n+1,1} \xrightarrow{\sim} S_{n+1,\xi}.$$

By G-equivariance of c_ξ, for any Schwartz function $\Phi \in \mathcal{S}(\mathfrak{s}_{n+1})$ and regular semi-simple $\gamma \in S_{n+1,\mathrm{rs},\xi}$, we have

$$\epsilon(\gamma)^{-1} \,\mathrm{Orb}(\gamma, c_{\xi,*}(\Phi)) = \varepsilon(c_\xi^{-1}(\gamma))^{-1} \,\mathrm{Orb}(c_\xi^{-1}(\gamma), \Phi).$$

Let $\lambda \in C_c^\infty(Q_{n+1,\xi})$ be any compactly supported function. Then $\mathrm{inv}^*(\lambda) \cdot c_{\xi,*}(\Phi)$ is a Schwartz function on S_{n+1} with support in $S_{n+1,\xi}$. The ratio

$$\varepsilon(c_\xi^{-1}(\gamma))/\epsilon(\gamma), \qquad \gamma \in S_{n+1,\xi}$$

is an algebraic invertible function by Lemma 5.3, and hence

$$\Psi_U := \frac{\varepsilon(c_\xi^{-1}(\gamma))}{\epsilon(\gamma)} \cdot \mathrm{inv}^*(\lambda) \cdot c_{\xi,*}(\Phi)$$

lies in $\mathcal{S}(S_{n+1})$ and satisfies

$$\mathrm{Orb}(\gamma, \Psi_U) = \lambda(\mathrm{inv}(\gamma)) \cdot \mathrm{Orb}(c_\xi^{-1}(\gamma), \Phi), \qquad \gamma \in \mathrm{inv}^{-1}(U). \tag{5.6}$$

We now make the following choices. Let $\Phi \in \mathcal{S}(\mathfrak{s}_{n+1})$ be the Gaussian test function from 4.15. Choose λ such that $\lambda \equiv c_{\xi,*}(e^{2\pi Q})$ on a neighborhood U of t. (Here, the quadratic form

$Q(y) = -\mathrm{tr}(y^2)$ has been descended to a function on Q_{n+1}.) Then (5.6) specializes to

$$\begin{aligned}\mathrm{Orb}(\gamma, \Psi_U) &= e^{2\pi Q(c_\xi^{-1}(\gamma))}\,\mathrm{Orb}(c_\xi^{-1}(\gamma), \Phi)\\ &= \begin{cases} 1 & \text{if } \gamma \in \mathrm{inv}^{-1}(U) \text{ matches to signature } (n,0)\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Here, in the final step, we have used the following lemma.

Lemma 5.4. *Let $y \in \mathfrak{s}_{n+1,\mathrm{rs},1}$ be a regular semi-simple element. Then y matches to signature (r, s) if and only if $c_\xi(\gamma)$ matches to signature (r, s).*

Proof. The Cayley transform can also be defined on the unitary side by the same formulas. The statement then follows from the definition of matching. □

This completes the proof of Theorem 5.1. □

Let us write $U(n+1)$ for $U(V_{(n,0)} \oplus \mathbb{C})$. For a smooth function ϕ on $U(n+1)$ and a regular semi-simple element $g \in U(n+1)_{\mathrm{rs}}$, we consider the $U(n)$-orbital integral

$$\mathrm{Orb}(g, \phi) = \int_{U(n)} \phi(h^{-1}gh)\, dh$$

where the Haar measure is normalized by $\mathrm{Vol}(U(n)) = 1$.

Corollary 5.5. *Let ϕ be an algebraic function on $U(n + 1)$, such as a matrix coefficient or the character of a finite-dimensional representation. Then there exists a Schwartz function $\psi \in \mathcal{S}(S_{n+1})$ such that for all $\gamma \in S_{n+1,\mathrm{rs}}$,*

$$\mathrm{Orb}(\gamma, \psi) = \begin{cases} \mathrm{Orb}(g, \phi) & \textit{if } \gamma \textit{ has a matching } g \in U(n+1)\\ 0 & \textit{otherwise.}\end{cases}.$$

Proof. Let $\mathbb{U}(n)$ and $\mathbb{U}(n+1)$ be the algebraic groups defining $U(n)$ and $U(n+1)$. The function ϕ being algebraic means that $\phi \in \mathbb{R}[\mathbb{U}(n+1)]$. The averaged function $\overline{\phi}(g) := \int_{U(n)} \phi(h^{-1}gh)$ then lies in the invariants $\mathbb{R}[\mathbb{U}(n+1)]^{U(n)}$. Since $\mathbb{U}(n)$ is connected, these are the same as the algebraic invariants $\mathbb{R}[\mathbb{U}(n+1)]^{\mathbb{U}(n)}$. In other words, $\overline{\phi}$ comes by pullback from the GIT quotient. In particular, there exists an algebraic function f on S_{n+1} such that $\overline{\phi} = \mathrm{inv}^*(f)$ and we have

$$\mathrm{Orb}(g, \phi) = f(\mathrm{inv}(g))$$

for all $g \in U(n+1)_{\mathrm{rs}}$.

Let Ψ be a Gaussian test function as in Theorem 5.1. Then, for every $\gamma \in S_{n+1,\mathrm{rs}}$, we find

$$\begin{aligned}\mathrm{Orb}(\gamma, f\Psi) &= f(\mathrm{inv}(\gamma))\,\mathrm{Orb}(\gamma, \Psi)\\ &= \begin{cases}\mathrm{Orb}(g,\phi) & \text{if } \gamma \text{ matches an element } g \in U(n+1)\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Thus $\psi = f\Psi$ satisfies the requirements of the corollary. □

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